Published in: David Bailey, Naomi Borwein, Richard Brent, Regina Burachik, Judy-Anne Osborn, Bailey Sims, Qiji Zhu (editors), *From Analysis to Visualization, A Celebration of the Life and Legacy of Jonathan M. Borwein, Callaghan, Australia, September 2017*, Springer-Verlag 2020, pp.133-139.

How mathematicians learned to stop worrying and love the computer

Keith Devlin, Stanford University, Stanford, CA 94305, USA

"How I learned to stop worrying and love the bomb", subtitle to the 1964 movie Dr. Strangelove

Abstract

Though mathematicians invented the modern computer as a theoretical entity, and a few of them helped build the first modern digital computers, mathematicians as a whole lagged far behind scientists, engineers, and other professionals in actually using them.[1] Recognizing that the field was in danger of falling too far behind, in May 1988, the American Mathematical Society launched a new section in its newsletter *Notices*, sent out to all members ten times a year, titled "Computers and Mathematics". Its aim was to promote the use of computers by mathematicians and provide them with information about the many new mathematical software systems being developed. The section was initially edited by the Stanford mathematician Jon Barwise, who ran it until February 1991, after which the AMS asked me to take it over. I held the reins from the March 1991 issue until the AMS and I decided to end the special section in December 1994. That six-and-a-half-year run achieved the intended goal. By the time the special section wound up, the computer had become a staple tool for mathematicians, both in teaching and research.

The modern, programmable, digital computer grew from theoretical mathematical results obtained in the 1930s and 40s by mathematicians such as Alan Turing (1912-54) in the UK and John von Neumann (1903-57) and Alonzo Church (1903-95) in the USA. Indeed, both Turing and von Neumann were involved in the design and construction of early digital computing devices for military purposes in their respective countries during the Second World War.[2]

[1] In this article, unless noted to the contrary, I use the term "mathematician" to refer to pure mathematicians, who focus on formulating and proving statements about abstract mathematical structures.

[2] While the results of Turing, von Neumann, and Church gave a theoretical underpinning to the subsequent developments of computers, it's clear that the technology would have been developed anyway, and indeed such advances were already underway. For example, Konrad Zuse took out patents for computing devices 1936 and 1941. And the ENIAC, 1943-46, was designed by engineers Eckert and Mauchly, before von Neumann became involved in the project. Moreover, theoretical and practical work on computing devices was done much earlier by Pascal (1642), Leibniz (1674), and Babbage (1822).

Yet, when digital computers became available for scientific work, starting in the 1950s, hardly any (pure) mathematicians made use of them. Indeed, that state of affairs continued through the 1960s and the 1970s, and well into the 1980s, a period in which the computer grew to be a ubiquitous tool in the natural sciences, in engineering, and the worlds of business and finance.

While mathematicians' seeming lack of interest in computers might have seemed strange—indeed paradoxical—to anyone outside the field, to those on the inside it was not at all surprising. The computer had little to offer the vast majority of research mathematicians.

There was no paradox here. Society's widespread layperson's assumption that mathematics is essentially "higher arithmetic" was always completely off base: (pure) mathematicians study abstract patterns and relationships.[3] For the most part, they use logical reasoning rather than numerical computation. Indeed, the early mathematical work on computing by Turing, von Neumann, Church, and others focused entirely on the theoretical *concept* of computation. Were it not for the demands of the war effort at the time, it is highly unlikely that Turing or von Neumann would ever have become involved in the design and construction of physical computing devices (Pilot ACE and the Manchester Mark I for Turing, ENIAC for von Neumann) and the execution of actual computations (code breaking in Turing's case and the calculation of artillery range tables and the design of the atomic bomb for von Neumann[4]).

To be sure, many applied mathematicians were quick to make use of the new technology, and specialized areas of mathematics such as numerical analysis grew considerably with the availability of computers. But the vast majority of mathematicians spent most of their time in day-to-day research activities that had remained largely unchanged for over two millennia. Their work progressed under a widespread, though unstated, assumption that computers could not possibly play a role in the construction of proofs of theorems.

That assumption was given a significant jolt in 1976, with the announcement by two mathematicians in the United States, the American Kenneth Appel and the German Wolfgang Haken, that they had made essential use of a computer to solve a famous, long standing open problem in mathematics: the Four Color Problem. Dating back to 1852, the problem had all the hallmarks of a theoretical problem for which a computer might be of no help. It asked for a proof that any map drawn on a plane can be colored using at most four colors so no two countries that share a stretch of border are colored the same. The answer will be yes or no; it is not about calculating a number.

If you try a few cases, say with maps of countries, states, or counties, you quickly start to believe the answer might be yes. But what about fictitious maps with thousands of regions, designed to require five or more colors? How do you deal with that possibility? Since there are infinitely many possible map configurations, it is not possible to prove the answer is yes by trying to color every possible one, even with a fast computer.

Or maybe the answer is no. A computer could perhaps be used to solve the problem in the negative; you could let the computer generate map after map and try to color them until it

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[3] The term "higher arithmetic" has acquired a special meaning in the mathematical world. That is not what I am referring to here.

[4] Making my title for this article a bit more than an irresistible play on words.

finds one that cannot be colored with four or fewer colors. Since the number of possible coloring configurations for a given map is finite, that could work. But if the answer to the problem is yes, that approach would never end. To solve the problem affirmatively, which is what the majority of mathematicians believed was the case, a logical argument would be required.

Attempts to solve the puzzle by many mathematicians over the years ended in failure, until Appel and Haken eventually came up with an approach that worked. Their approach did indeed involve a logical argument, but on its own that argument did not solve the problem. Rather, they were able to show that if every map in a specific collection of 1,476 particular map configurations could be colored with at most four colors, then the same would be true for all maps. The two researchers then wrote a computer program to examine all possible four-color coloring schemes for those 1,476 maps in turn to see if, in each case, it could find one that worked for that map. That task would have taken too long for a human to complete, even a team of humans, but their computer completed the task in a few months. (Today's computers could do it much faster.) The computer search proved successful, and the Four Color *Problem* became the Four Color *Theorem*. Mathematics had entered a new era.

Initially, the Appel and Haken proof generated a considerable amount of controversy among mathematicians, many of whom regarded the use of a computer to prove a theorem in the same way sports fans object to the use of performance enhancing drugs. But when, over the ensuing years, a number of other theorems were proved using arguments that likewise required use of a computer, the objections gradually died down. The writing was clearly on the wall—or rather, on the computer screen. As in many other walks of life, for mathematics, the computer was here to stay.

Even so, for the vast majority of mathematicians, things remained the same. Proving a new result still required the construction of a suitable logical argument. The only new twist was that it became accepted that the argument might, on occasion, depend on the successful execution of a computation (often an exhaustive search through a large but finite sets of possibilities), and such arguments were accepted as legitimate proofs. Referees of papers submitted for publication would check the logic in the traditional way and either take the computation on trust or, if feasible, arrange for an independently written computer program running on a different computer to check that the computational part did as the authors claimed.

Notable examples of computer-assisted proofs, as they became known, are:

- Proof of Feigenbaum's universality conjecture in non-linear dynamics (1982)
- Proof of the non-existence of a finite projective plane of order 10 (1989)
- Proof of the Robbins conjecture (1996)
- Proof of Kepler's sphere packing conjecture (1998).

There were also cases where computers were used to establish negative results; for example, Odlyzko and te Riele's disproof of the Mertens Conjecture (1985). But since such examples were rare, mathematicians by and large continued doing business as usual. The only time they used a computer were for email, after it was introduced in the 1980s, and for typing manuscripts. As things turned out, that latter use of computers for manuscript preparation provided the final impetus that resulted in the American mathematical community embracing the new technology for teaching and research.

In 1978, the Stanford mathematician and computer scientist Donald Knuth released the first version of his mathematical typesetting system TeX, which enabled mathematicians to type their own books and papers using a regular computer keyboard. Special commands were used to product Greek letters and mathematical symbols, and the program took care of organizing the layout on the page so that it was both mathematically correct and aesthetically pleasing. There was a fairly steep learning curve as a new user mastered the typesetting language, which was made somewhat easier with the appearance in the early 1980s of LaTeX, a more user-friendly front-end package for TeX, developed by Leslie Lamport of SRI.

So great and so obvious were the benefits of using LaTeX, that some mathematicians quickly adopted it, but even with LaTeX there was still a significant learning curve, any many were put off. They could still see the advantages of typing their own manuscripts, however, and so they went with one of a number of what-you-see-is-what-you-get, mathematical word-processing systems that offered drop down menus of alternative alphabets and mathematical symbols, an approach that was much easier to learn but did not produce the elegant page layout you got from TeX.

In 1987, Richard Palais of Brandeis University wrote a series of articles for the American Mathematical Society *Notices*, surveying for mathematicians the various mathematical word processing systems that were available at the time. The interest in those articles was sufficiently strong for mathematicians at the AMS to start talking about the Society taking a pro-active role in helping the community take advantage of the new working possibilities that computers were starting to offer, not only in preparing manuscripts but in teaching and research. That led to a decision for the *Notices*, which was sent to all members ten times a year, to introduce a regular section "Computers and Mathematics", that would serve both to provide inspiration for mathematicians to make greater use of computers, and to act as an information exchange for the various possibilities computers offered in their work.

That same year, 1987, was also when I moved from the UK to the United States, to spend a year as a Visiting Professor at Stanford. My host, Jon Barwise, was the mathematician the AMS asked to edit the new *Notices* section, and the two of us talked about the upcoming new column on a number of occasions.

As mathematicians, we both had spectators' interest in the use of computers within traditional mathematics—indeed, Jon attended the lavish event launching Steve Wolfram's new mathematical software system *Mathematica* on June 23, 1988— but our main interests took different forms. Jon's interest was primarily that of a logician, and he soon began working with his Stanford colleague John Etchemendy to develop instructional software to teach formal logic (*Turing's World*, *Tarski's World*, and *Hyperproof*). My focus was more as part of my growing interest in what would become known as mathematical cognition, where the focus was on studying mathematics as a mental tool, looking at how it arose, and how it related to, fitted in with, and complemented other forms of thinking. From that standpoint, the use of computers to assist in doing mathematics was but one component of what I would end up calling "mathematical thinking".

The "Computers and Mathematics" section launched in the May/June 1988 issue of the *Notices*, with Barwise leading off with an essay in which he declared that the goal was to reflect, both practically and philosophically, on cases where computers were affecting mathematicians and how they might do so in the future; to act as an information exchange into what software products were available; and to publish mathematicians' reviews of new software.

Barwise edited the section through to February 1991, after which the AMS asked me to take it over. I held the reins from the March 1991 issue until the AMS and I decided to end the special section in December 1994. The reason? That six-and-a-half-year run of the special section had achieved the intended goal. The computer had become a staple tool for mathematicians, both in teaching and research.

The general format of each column was to start with some form of editorial comment, then, frequently, a feature article solicited by the editor, and then a number of reviews of new mathematical software. In all, we published 59 feature articles, 19 editorial essays, and 115 reviews of mathematical software packages (31 features, 11 editorials, and 41 reviews under Barwise, 28 features, 8 editorials, and 74 reviews under me).

At around the same time the "Computers and Mathematics" section was starting up, a number of mathematicians were developing a new subfield of mathematics called "Experimental Mathematics". In this field, one of the primary goals in using computers was to formulate conjectures that could subsequently by proved by conventional means—which cast the computer as an additional weapon in the pure mathematician's armory rather than a completely separate technological endeavor. In 1992, a new journal with that name as its title was established by the American mathematicians David Epstein, Silvio Levy, and the German-American mathematics publisher Klaus Peters. And in the fall of that year, the Canadian mathematicians Jonathan and Peter Borwein sent me their article "Some Observations on Computer Aided Analysis", written to introduce their new field to the mathematical community at large, which I published in the October issue of "Computers and Mathematics".

At the same time as the computer was starting to change mathematics research and applications, various instructors brought it into their classrooms. Computer Algebra Systems such as *Mathematica* and *Maple* were used to teach calculus in a new way, and a number of new textbooks to support such teaching came onto the market. Some of the articles and product reviews in "Computers and Mathematics" were devoted to that increasing use of computers in the world of university mathematics education. Things were starting to move very quickly.

When he introduced the last section he edited, Barwise had written:

> "Whether we like it or not, computers are changing the face of mathematics in radical ways, from research, to teaching, to writing, personal communication, and publication. Over the past couple of years we have seen numerous articles about these developments.
>
> Computers are even forcing us to expand our idea about what constitutes doing mathematics, by making us take much more seriously the role of experimentation in mathematics. (I draw attention to a new journal devoted to experimental mathematics below.)
>
> One view of the future is that mathematics will come to have (or already has) two distinct sides: experimentation, which can exploit the speed and graphics abilities of programs like

Maple and *Mathematica*, to allow us to spot regularities and make conjectures, and proof, very much in the style of today's mathematics. …

Whether we applaud or abhor all these changes in mathematics, there is no denying them by turning back the clock, anymore than there is in the rest of life. Computers are here to stay, just as writing is, and they are changing our subject."

It is surely obvious from those final remarks that the computer was seen as something of a threat by some mathematicians, and the "Computers and Mathematics" section was not without its detractors.

Taking over a month later, I began by saying that:

"This column is surely just a passing fad that will die away before long. Not because mathematics will cease to have much connection with computers, but rather, quite the reverse: the use of computers by mathematicians will become so commonplace that no one thinks to mention it any more."

When I wrapped up the section four years later, I wrote:

"With its midwifery role clearly coming to an end, the time was surely drawing near when "Computers and Mathematics" should come to an end. The change in format of the *Notices*, which will take place at the end of this year, offered an obvious juncture to wind up the column. …

The disappearance of this column does not mean that the *Notices* will stop publishing articles on the use of computers in mathematics. Rather, recognizing that the use of computer technology is now just one more aspect of mathematics, the new *Notices* will no longer single out computer use for special attention. I'll drink to that.

The child has come of age."

And so mathematics moved on. In 2004, Jon Borwein and David Bailey published (together with co-authors in two cases) the first of what would be three major research monographs on experimental mathematics, and in 2008, Jon and I published our expository text *The Computer as Crucible: An Introduction to Experimental Mathematics*. A year later, Wolfram released *Wolfram Alpha*, an online computational tool that, among other things, was able to execute practically any mathematical method or procedure—faster and more accurately than any human, and with effectively no restrictions on data size.

The computer had, by then, completely revolutionized all of procedural mathematics. Only the pure mathematicians, who focus on finding proofs of precisely worded theorems, remained almost entirely unscathed by the revolution.

In late 2016, after I learned of Jon's tragic early passing, I looked back on my own mathematical work in the twenty years after I edited my last *Notices* "Computers and Mathematics" section, some of it with Jon. My reflections prompted me to pen—more accurately type (on a computer)—an opinion piece for the *Huffington Post*, which was published on January 1, 2017, with the startling, but absolutely accurate title: "All the Mathematical Methods I Learned in My University Math Degree Became Obsolete in My Lifetime". For the fact is, that over a period of just under a quarter-century, during which time I moved from working in pure mathematics (i.e., focusing on proofs) to making use of mathematics to solve large-scale, real-world problems, my daily experience of doing mathematics changed from using methods and executing procedures to putting problems into a form where I could apply a powerful computational tool such as (in my case) *Wolfram Alpha* or *Mathematica*.[5]

True, by then I was no longer a pure mathematician, so my experience here is not typical of pure math. But it ***is*** typical of the way doing math has changed for the vast majority of mathematicians in the world. Besides, no one can look at the computer-intensive work of Jonathan Borwein and David Bailey in Number Theory, where they also use *Mathematica*, and pretend it is anything other than pure math. To be sure, some pure mathematicians make hardly any professional use of computers aside from email and an occasional Google search. But for a great many, the computer is now an integral part of how they carry out their work.

That then, is the story of how mathematicians learned to stop worrying and love the computer. I could go on, and dig much deeper into the details. But, given the ease with which, given a few key issues (and associated key words) we can now all dig down on our own, I'll let you get a sense of the mathematical computer revolution by browsing the image gallery that accompanies this short article.

References

For a complete index to everything published in the "Computers and Mathematics" section of the AMS *Notices*, see Keith Devlin and Nancy Wilson, *Notices of the American Mathematical Society* Vol 42, No. 2, February 1995, 248–254, http://www.ams.org/journals/notices/199502/devlinsixyear.pdf

Accessible books the reader may find helpful are: *Four Colors Suffice: How the Map Problem Was Solved*, by Robin Wilson (2003), *The Computer as Crucible*, by Jon Borwein and myself (2008), *Turing's Cathedral*, by George Dyson (2012), and *Computing: A Concise History*, by Paul Ceruzzi (2012).

[5] Full disclosure. I was a member of Wolfram's initial *Mathematica* Advisory Board in the product's early years (we were all unpaid), so I naturally defaulted to using Wolfram products. But there were several CASs being developed around the same time, *Maple*, *Matlab*, *Magma*, *Sage*, etc.

APPENDIX

Proceedings of the Jon Borwein Commemorative Conference, 2017

How mathematicians learned to stop worrying and love the computer

IMAGE GALLERY

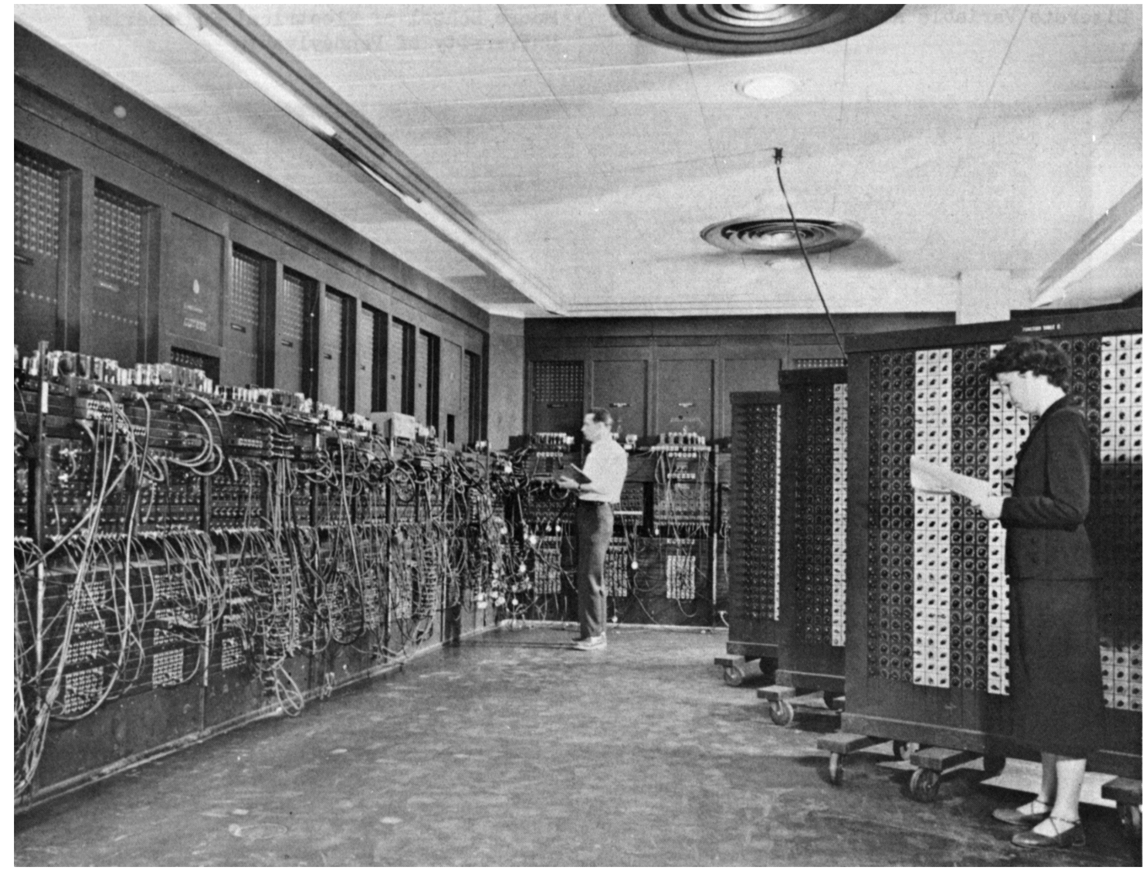
Keith Devlin, Stanford University, Stanford, CA 94305, USA

The two most iconic computer pioneers

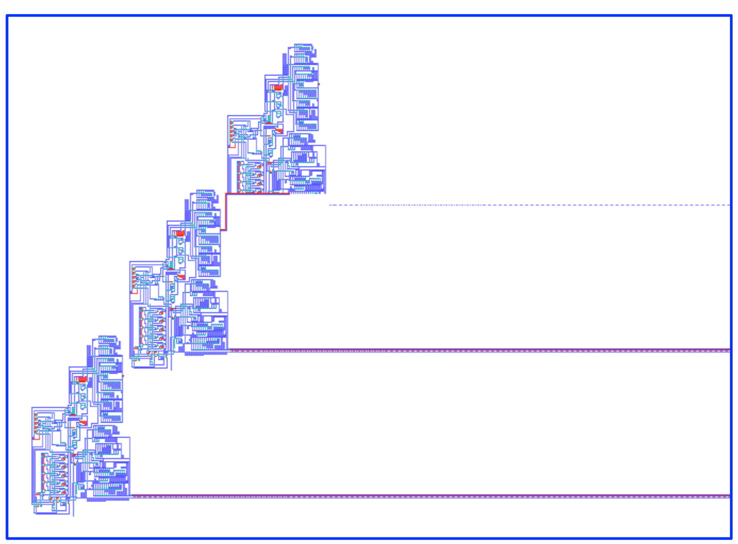
John von Neumann (1903-57)

Alan Turing (1912-54)

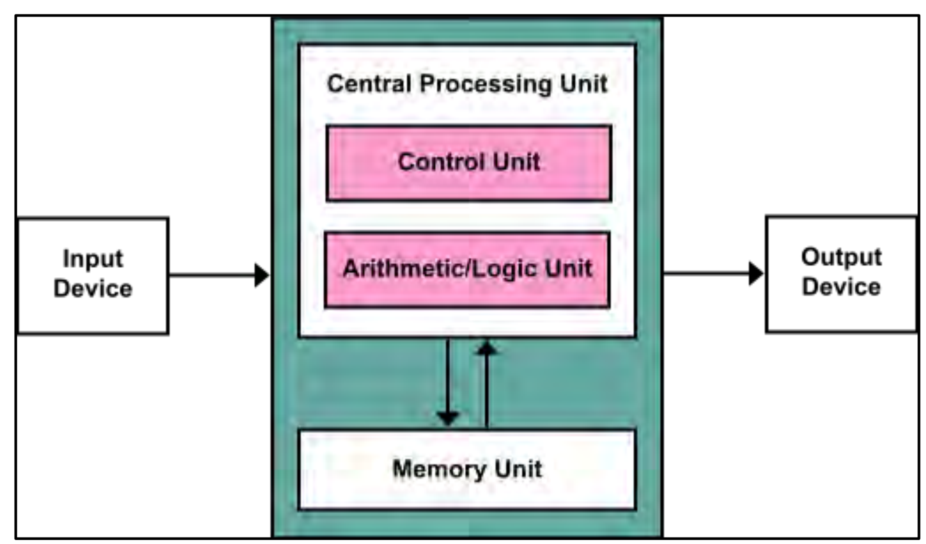
von Neumann - from theory to practice



ENIAC: Electronic Numerical Integrator and Computer (1946)
Ballistic Research Laboratory (BRL), Aberdeen Proving Ground, MD

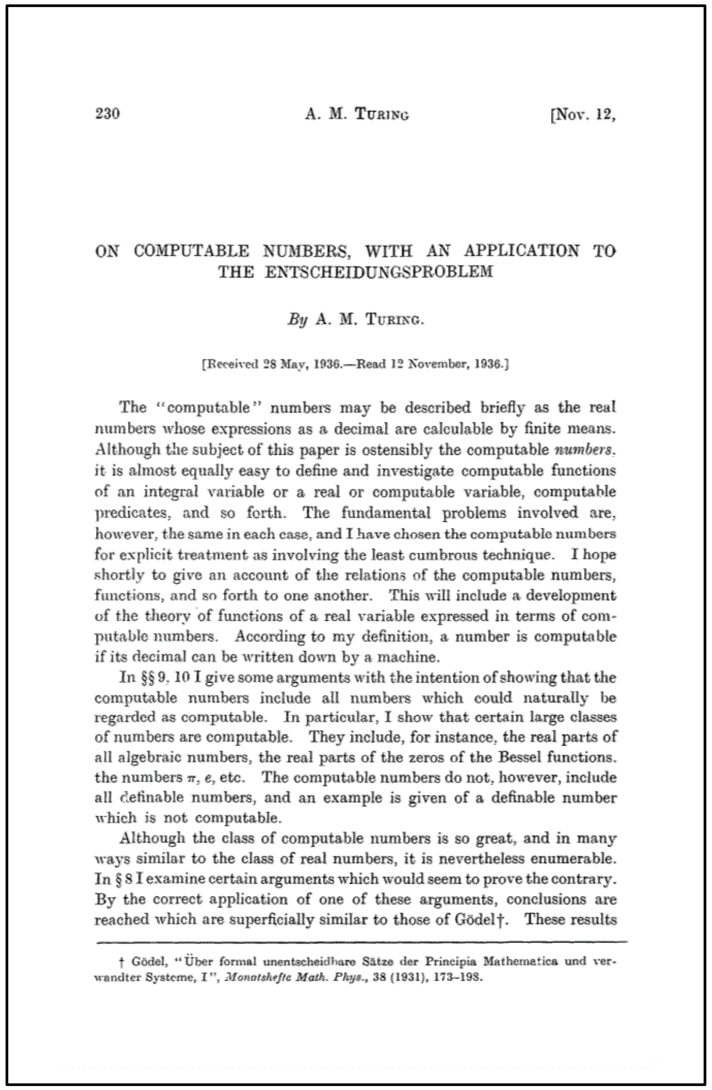


Cellular automata (1940s)



von Neumann architecture (1945)

Turing - from theory to practice



230 A. M. TURING [Nov. 12,

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. TURING.

[Received 28 May, 1936.—Read 12 November, 1936.]

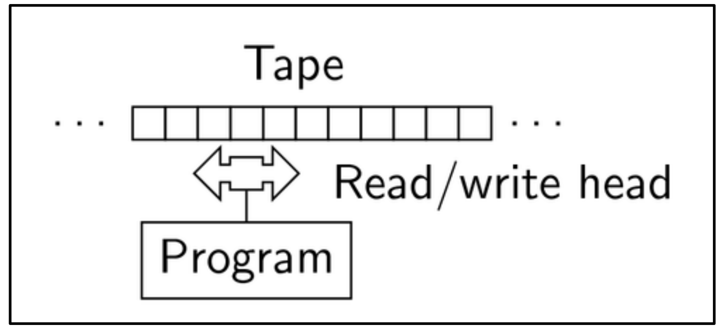
The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

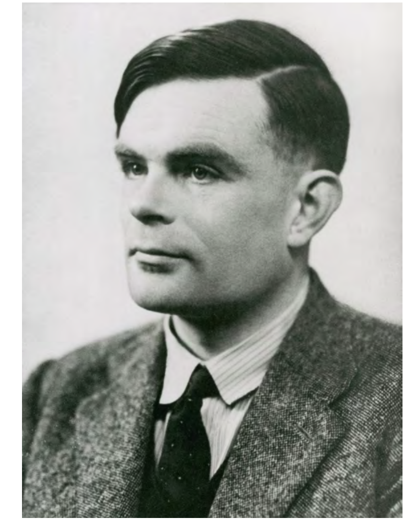
In §§ 9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers π, e, etc. The computable numbers do not, however, include all definable numbers, and an example is given of a definable number which is not computable.

Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless enumerable. In § 8 I examine certain arguments which would seem to prove the contrary. By the correct application of one of these arguments, conclusions are reached which are superficially similar to those of Gödel†. These results

† Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I", *Monatshefte Math. Phys.*, 38 (1931), 173–198.

Proc. of the London Math. Soc.
Series 2, 42 (1936), pp 230–265





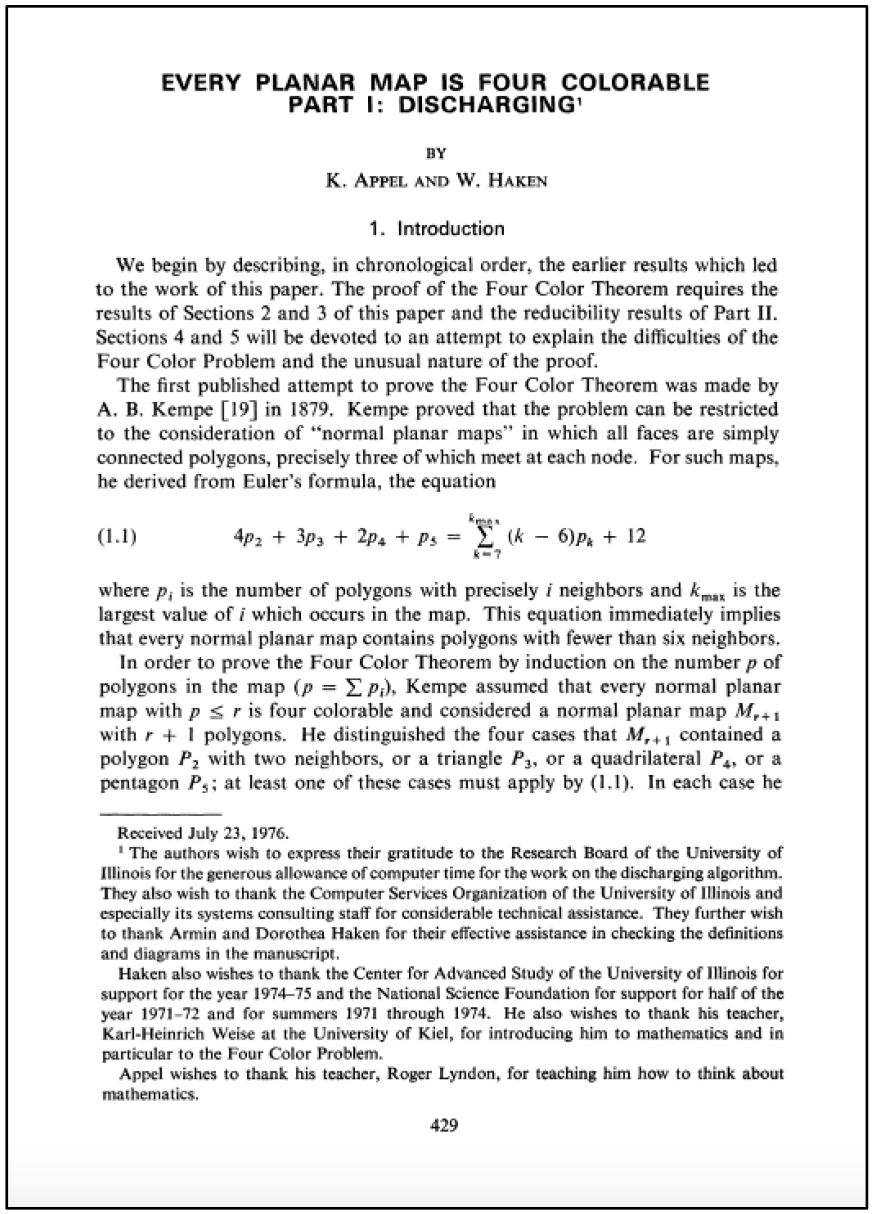
Tape symbol	Current state A			Current state B			Current state C		
	Write symbol	Move tape	Next state	Write symbol	Move tape	Next state	Write symbol	Move tape	Next state
0	P	R	**B**	P	L	**A**	P	L	**B**
1	P	L	**C**	P	R	**B**	P	R	**HALT**

Turing Machine (1936)

Manchester Mark I (1949)

A SHOCK: The Four Color Theorem (1976)

1852, Francis Guthrie asked: Can every possible map be colored using at most 4 colors?



EVERY PLANAR MAP IS FOUR COLORABLE
PART I: DISCHARGING[1]

BY

K. APPEL AND W. HAKEN

1. Introduction

We begin by describing, in chronological order, the earlier results which led to the work of this paper. The proof of the Four Color Theorem requires the results of Sections 2 and 3 of this paper and the reducibility results of Part II. Sections 4 and 5 will be devoted to an attempt to explain the difficulties of the Four Color Problem and the unusual nature of the proof.

The first published attempt to prove the Four Color Theorem was made by A. B. Kempe [19] in 1879. Kempe proved that the problem can be restricted to the consideration of "normal planar maps" in which all faces are simply connected polygons, precisely three of which meet at each node. For such maps, he derived from Euler's formula, the equation

$$4p_2 + 3p_3 + 2p_4 + p_5 = \sum_{k=7}^{k_{max}} (k-6)p_k + 12 \qquad (1.1)$$

where p_i is the number of polygons with precisely i neighbors and k_{max} is the largest value of i which occurs in the map. This equation immediately implies that every normal planar map contains polygons with fewer than six neighbors.

In order to prove the Four Color Theorem by induction on the number p of polygons in the map ($p = \sum p_i$), Kempe assumed that every normal planar map with $p \leq r$ is four colorable and considered a normal planar map M_{r+1} with $r + 1$ polygons. He distinguished the four cases that M_{r+1} contained a polygon P_2 with two neighbors, or a triangle P_3, or a quadrilateral P_4, or a pentagon P_5; at least one of these cases must apply by (1.1). In each case he

Received July 23, 1976.

[1] The authors wish to express their gratitude to the Research Board of the University of Illinois for the generous allowance of computer time for the work on the discharging algorithm. They also wish to thank the Computer Services Organization of the University of Illinois and especially its systems consulting staff for considerable technical assistance. They further wish to thank Armin and Dorothea Haken for their effective assistance in checking the definitions and diagrams in the manuscript.

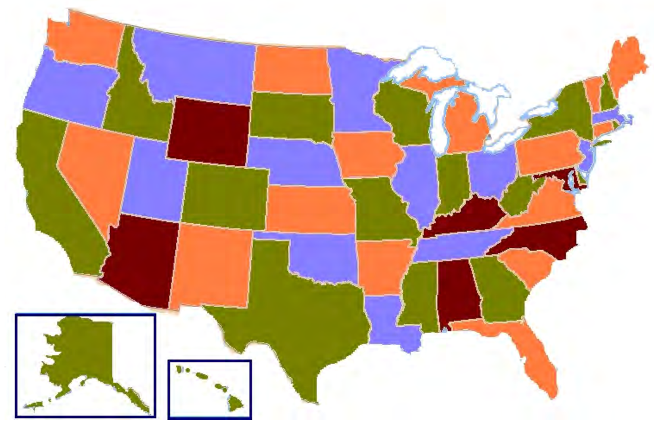
Haken also wishes to thank the Center for Advanced Study of the University of Illinois for support for the year 1974–75 and the National Science Foundation for support for half of the year 1971–72 and for summers 1971 through 1974. He also wishes to thank his teacher, Karl-Heinrich Weise at the University of Kiel, for introducing him to mathematics and in particular to the Four Color Problem.

Appel wishes to thank his teacher, Roger Lyndon, for teaching him how to think about mathematics.

429

Illinois J. Math. 21 (1977), 429-490

[PART II: Reducibility, pp.491-567]



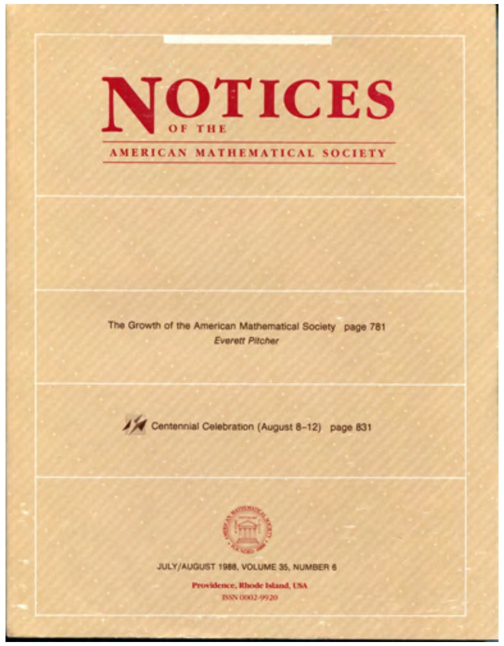
A four coloring of the United States

1976, proved by Kenneth Appel and Wolfgang Haken

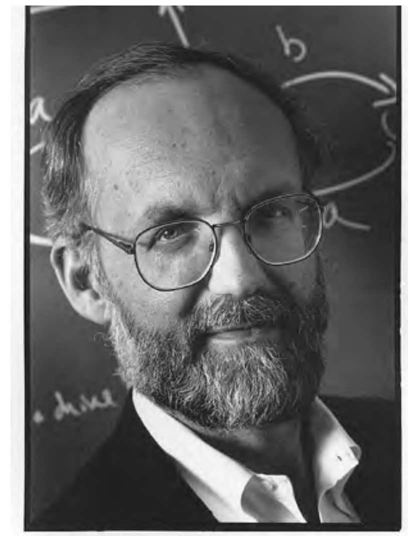
Proof made essential use of a computer to check 1,476 special map configurations

K. Appel & W. Haken

The AMS Notices "Computers and Mathematics" column



Over its six-and-a-half years run, the column published 59 feature articles, 19 editorial essays, and 115 reviews of mathematical software packages
— 31 features 11 editorials, and 41 reviews under Barwise, 28 features, 8 editorials, and 74 reviews under Devlin.


Jon Barwise (1942-2000)

SECTION LAUNCH

FEATURE COLUMNS

690 Inside the AMS: Report of the Treasurer (1987)

The annual report includes a review of the Society's operations during the past year.

693 Computers and Mathematics *Jon Barwise*

A new column is introduced, designed to examine the increased interaction between computers and mathematics. In this first feature, a noted mathematician ponders the impact of computers on his work and that of other mathematicians.

NAMS, May/June 1988, from the title page

In his introductory essay, Barwise declared that the goal of the new column was to reflect, both practically and philosophically, on cases where computers were affecting mathematicians and how they might do so in the future; to act as an information exchange into what software products were available; and to publish mathematicians' reviews of new software.

Feb 1991: Barwise bows out of "Computers and Mathematics"

with farewell reflections and a new journal announcement

The Changing Face of Mathematics

Whether we like it or not, computers are changing the face of mathematics in radical ways, from research, to teaching, to writing, personal communication, and publication. Over the past couple of years we have seen numerous articles about these developments.

Computers are even forcing us to expand our ideas about what constitutes doing mathematics, by making us take much more seriously the role of experimentation in mathematics. (I draw attention to a new journal devoted to experimental mathematics below.)

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... ...

Whether we applaud or abhor all these changes in mathematics, there is no denying them by turning back the clock, anymore than there is in the rest of life. Computers are here to stay, just as writing is, and they are changing our subject. At least this is my firm conviction, and it is the reason I have been willing to serve as editor of this column for something over two years.

The Journal of Experimental Mathematics

One of the topics that has come up several times over the past couple of years in this column is the development of experimental techniques in mathematics, both for the discovery of new insights and conjectures. A new journal in this area has now been formed devoted to the publication of experimental mathematics. Part of the draft announcement I have seen reads "It is hoped that the journal will help generate a climate in which accounts of interesting experiments are not confined to private notebooks and suppressed from accounts of the mathematics they inspire, but see the light of day, to the benefit of researchers, students, and the mathematical community in general."

Is this to be taken seriously? If you have any doubts, the list of editors and advisors to the journal should be interesting: F. Almgren, H. Cohen, R. Devaney, D. Epstein (Editor-in-chief), R. Graham, D. Hoffman, H.W. Lenstra, S. Levy, R. Llave, B. Mandelbrot, A. Marden, D. Mumford, U. Pinkall, P. Sarnak, J.P. Serre, and W. Thurston.

As the above quote and this list make clear, what we are talking about is nothing less than a possible revolution in the way mathematicians think about and report their work. Anyone familiar with the standards of experimentation in other branches of science can only wonder what standards will evolve for judging experimental work in mathematics. It is an exciting event.

Mar 1991: Devlin takes over “Computers and Mathematics”

starting with a look to the future (short for the column, long for the computer in mathematics)

Why “Computers and Mathematics?”

This column is surely just a passing fad that will die away before long. Not because mathematics will cease to have much connection with computers, but rather, quite the reverse: the use of computers by mathematicians will become so commonplace that no one thinks to mention it any more.

As far as the use of scientific text processors to write papers and books is concerned, that state of affairs is probably here already, or at least very close, with Donald Knuth's TEX clearly the favored tool. As J. I. Hall of Michigan State University reported in this column in January, “virtually all of the larger math departments which responded to the survey have converted their technical typing staff to TEX, in one of its many configurations.” And yet it was only four years ago, in 1987, that Richard Palais of Brandeis University organized the series of articles in *Notices*, describing the various mathematical word processing systems available, that for many of us was the first real introduction to the range of products becoming available for the preparation of mathematical documents. And I suspect that for most of us it

... ...

Imagine then the kind of person coming into our graduate schools, if not today, then certainly tomorrow. Brought up from early childhood on a diet involving MTV, Nintendo, graphical calculators packed with algorithms, Macintosh-style computers, and, in the not-too-distant future, hypermedia educational tools as well. Such a person is going to enter mathematics with an outlook and a range of mental abilities quite different from their instructors—in fact I see no *a priori* reason why they should be the same people who would have become successful mathematicians had they come along a generation earlier.* Such a profound change in outlook and skills, and probably also personnel-type, will surely send mathematics into directions few of us can presently foresee.

Just before Christmas, I was discussing a particular proof with one of the sophomore undergraduates in my course on abstraction and proofs. Much to my surprise, the way the student described the entire proof process, in a quite matter of fact way, was as the unification (or amalgamation) of different pieces of information. (He did not use the word “unification,” but he did keep referring to “information” and the merging of different information.) The *linearity* of the proof, which always seemed so important to me when I was a student, hardly came into the picture at all as far as this young mathematician was concerned.

Nov/Dec 1994: The column closes down

Coming of Age

"This column is surely just a passing fad that will die away before long. Not because mathematics will cease to have much connection with computers, but rather, quite the reverse: the use of computers by mathematicians will become so commonplace that no one thinks to mention it anymore."

The above paragraph opened my introductory editorial for this column in the March 1991 issue of the *Notices*, when I took over the stewardship of the column from founding editor Jon Barwise. Over the three and a half years I have edited the column, I have continued to view its very existence with ambivalence.

Though it must be obvious to all that the development of the modern computer has changed mathematics forever, both in terms of content (the addition of new areas of research, not the elimination or replacement of any old areas) and the way many of us go about our daily lives as mathematicians, I always felt that there was altogether too much hype. I looked forward to the day when, for most of us, the computer settled back to become an accepted part of the mathematician's working environment, ranked alongside the blackboard, the pencil and scratch pad, the telephone, and the coffee pot. We are in the business of mathematics, not computing. Editing a column entitled "Computers and Mathematics", the function of which was to concentrate entirely on the computer connection, did not seem consistent with my view of the computer as simply one of the tools we use to get the job done.

... ...

On the other hand, the column clearly filled a definite need within the mathematical community, as mathematicians learned how to take advantage of the new, electronic tools being made available—I soon lost count of the number of mathematicians who, on meeting me, would say how valuable they found the column, with its mixture of feature articles and software reviews and the occasional editorial comment from the column editor (the original format developed by Barwise).

I guess I ended up like the newly elected president who declares that he did not seek the office, nor did he agree with the system that elected him, but he would serve anyway, for the general good. Not that the task was painful. Or difficult. I was, and still am, interested not only in mathematics and computing but—as a logician—how the latter effects the way we do the former and how the former supports the latter.

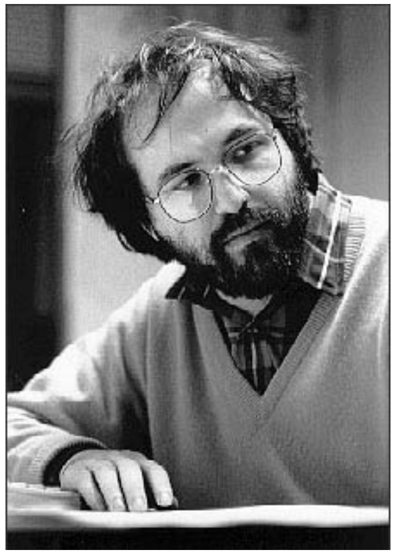
Things have moved swiftly in the mathematics community. Possibly the last scientists to avail themselves of computer technology, in just a few short years mathematicians have, for the most part, embraced it in a significant and far-reaching way, with the result that the mathematics community is now one of the leading users of computer technology.

Nov/Dec 1994: The column closes down

With its midwifery role clearly coming to an end, the time was surely drawing near when "Computers and Mathematics" should come to an end. The change in format of the *Notices*, which will take place at the end of this year, offered an obvious juncture to wind up the column. Thus, this will be the final regularly scheduled "Computers and Mathematics" section in the *Notices*. The February issue will provide a reference bibliography of all the articles and reviews that have appeared in the column since its inception—something many readers have asked for on numerous occasions.

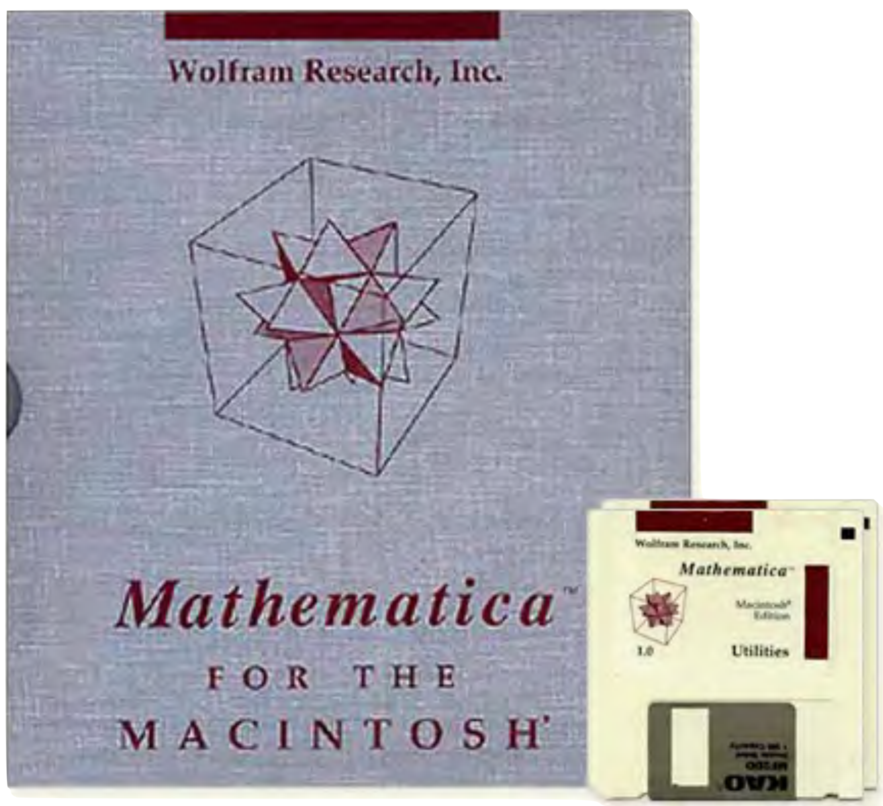
The disappearance of this column does not mean that the *Notices* will stop publishing articles on the use of computers in mathematics. Rather, recognizing that the use of computer technology is now just one more aspect of mathematics, the new-look *Notices* will no longer single out computer use for special attention. I'll drink to that.

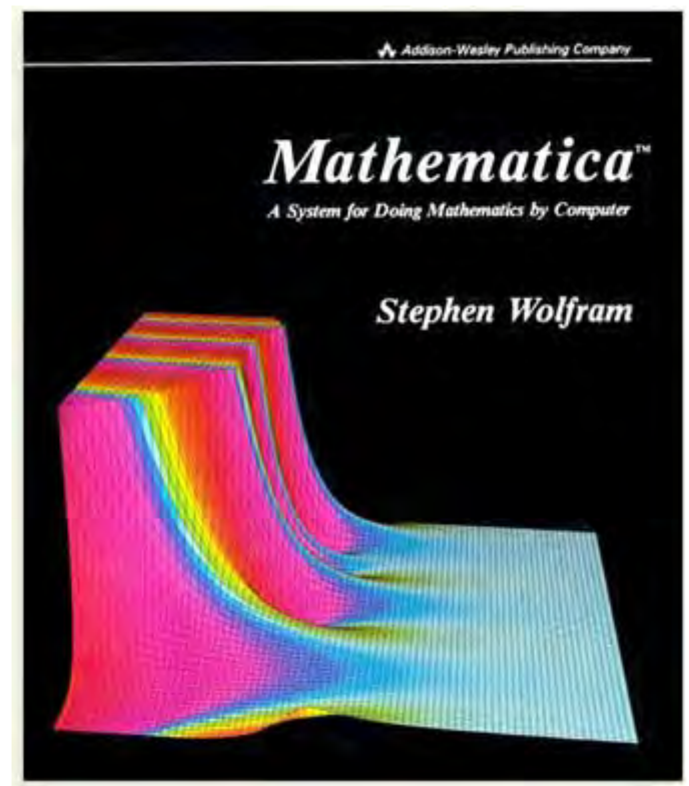
The child has come of age.



1988: Stephen Wolfram's *Mathematica*

v1.0 released June 23, 1988





Mathematica launch

Corporate Speakers at the *MATHEMATICA*™ Product Announcement (June 23, 1988):

- **Forest Baskett**, Vice President, Research & Development, Silicon Graphics Computer Systems
- **Gordon Bell**, Vice President, Research, Development and Engineering, Ardent Computer
- **Steven Jobs**, President, NeXT, Inc.
- **William Joy**, Vice President, Research & Development, Sun Microsystems
- **Vicky Markstein**, Research Staff Member, IBM
- **Eric Lyons**, Director of Technology, Autodesk, Inc.
- **Larry Tesler**, Vice President, Advanced Technology, Apple Computer
- **Stephen Wolfram**, President, Wolfram Research, Inc., and Professor of Physics, Mathematics and Computer Science, University of Illinois

Bundled with NeXT

Media interest in *Mathematica*

3.14159265358979323846264338327950288419716939937510

RESOURCES

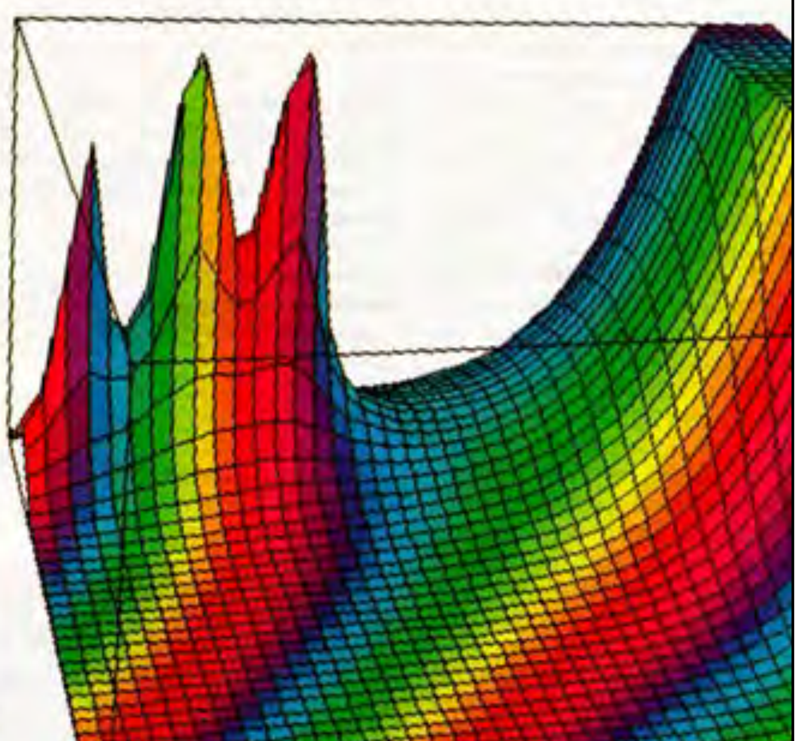
Enter Mathematica

In which a few thousand years of mathematical knowledge suddenly arrive on the desktop, and we journey through a strange and wonderful new world

Remember when you were in high school in that (expletive deleted) math class, and you couldn't understand the equation you had to solve? Wouldn't it have helped to be able to type the equation on your Mac and instantly see a picture of it — and a solution? What if you could check similar equations quickly to see how they differed? With Mathematica, an astoundingly powerful new mathematics program from Wolfram Research, such feats become mere child's play.

Mathematica opens up the entire world of mathematics to exploration by both the curious and the professional. What a spreadsheet can do with arithmetic, Mathematica can do with all of mathematics. Combining the knowledge of a mathematics Ph.D. with the speed of a computer, Mathematica is a breakthrough product in the world of small computers. It's powerful enough to provide professional physicists, mathematicians, or engineers with important new tools, yet simple enough to help students with their algebra.

Mathematica performs numeric calculations to any desired degree of precision. It does symbolic mathematics, manipulating algebraic formulae with ease. It produces dazzling two-



BY JAMES FINN

Information Processing

SOFTWARE

IT'S HOT, IT'S SEXY, IT'S ... CALCULUS?

Stephen Wolfram's Mathematica may create a giant market

When Stephen Wolfram taught an introductory physics class to undergraduates at the University of Illinois last spring, he was appalled by the display of apathy. "The students didn't give a damn," he recalls. "It was very depressing." That convinced him that he had done the right thing in 1987, when he set up his own business partly because he thought it might leave a bigger imprint on the world of mathematics than he could in a lifetime of teaching.

A year later a lot of customers agree with Wolfram. Since founding Wolfram Research Inc., he has lined up an impressive list of licensees: IBM, Sun Microsystems, Silicon Graphics, and Sony are selling or planning to sell a version of his product, called Mathematica, for their computers. Steven P. Jobs, who persuaded Wolfram to use the name Mathematica, is including the program with each of his Next Inc. machines. And Wolfram Research sells a $500 version for the Apple Macintosh. "That basically makes Mathematica the standard," says Steven M. Christensen, a research scientist at the National Center for Supercomputing Applications at the University of Illinois in Urbana-Champaign. Some analysts have likened Mathematica to Lotus Development Corp.'s 1-2-3 spreadsheet program, pegging the potential market for Mathematica at $500 million.

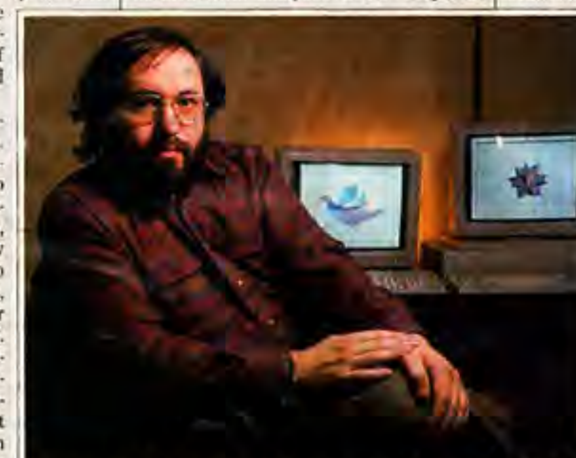
MATH APPEAL. Mathematica is an extraordinarily powerful calculator. But unlike simple calculation programs, it does more sophisticated math as well. Not only does it do arithmetic grunt work such as finding logarithms, but it can also do calculus and solve differential equations and algebra problems. Moreover, the program helps students visualize mathematical functions by displaying them in intricate and colorful pictures. "It's going to revolutionize mathematics instruction," declares Curtis S. Wozniak, general manager of the education products division at Sun, whose version of Mathematica is expected to be shipped next month. Wozniak says that 20% of Sun's business is in education.

Mathematica is particularly useful for scientists and anyone doing serious mathematical analysis. G-Bar Corp., an arbitrage firm in Chicago, uses the program to do stock market analysis. Christensen, a theoretical physicist, says Mathematica helped him solve complex equations in a few days instead of the months it would have taken him to write his own program.

Despite a late start, Wolfram has the potential to take a quick market lead. Symbolics Inc. in Cambridge, Mass., the University of Waterloo in Ontario, and a few others have been selling theoretical math programs for workstations and minicomputers since the early 1980s. And PC software firms such as Universal Technical Systems Inc. in Rockford, Ill., and Mathsoft Inc. in Cambridge sell more limited equation solvers. Mathematica could appeal to a larger market, as it performs both basic calculations and high-level mathematics. Mathsoft's new product for Sun Microsystems Inc.'s workstations, called MathStation, comes the closest to direct competition.

What's more, Mathematica appears to be the right program at the right time. As PCs become more powerful, they can more easily run programs such as Mathematica, which requires at least 2.5 megabytes of internal memory. Predicts Fred Thorlin, an analyst at Dataquest Inc. in San Jose, Calif.: "It won't happen overnight, but Mathematica could be stupendously successful."

'GENIUS' GRANT. A native of London, Wolfram, 29, started early, writing research papers on particle physics while still at Eton, the elite private school near London. He didn't graduate, but that didn't stop him from entering Oxford University at age 17. Without a degree from Oxford, he entered the California Institute of Technology as a graduate student. A few weeks after his 20th birthday, he received his PhD. At age 21, Wolfram was the youngest recipient of a John D. and Catherine T. MacArthur Foundation "genius" grant—$125,000 over five years to use as he pleased. He used some of that and most of his savings, about $250,000 in all, to start Wolfram Research.



WOLFRAM: HIS SOFTWARE COULD REVOLUTIONIZE MATH INSTRUCTION

Wolfram was just as precocious in his business dealings. With some fellow students at Caltech, he developed SMP, a mathematics program for use by research scientists. Sensing an opportunity, Wolfram wanted to license SMP to a company to sell. But a protracted and bitter struggle ensued, and in 1982 Caltech won ownership of the program, eventually selling the distribution rights on its own. Soon afterward the disillusioned Wolfram left Caltech for Princeton University and then the University of Illinois, where he's a tenured professor of physics, mathematics, and computer science.

It took Wolfram and seven colleagues a year to design Mathematica. The company now has about 35 employees, with Wolfram as president and CEO. His goal is to see his program become as indispensable to PCs as word processors are. Then, he says, he will know he has made his mark. "Writing an academic paper about your computer program isn't the way to make a difference," Wolfram says. "It's a lot more exciting to make something a lot of other people can use." With so many computer manufacturers already in his corner, the odds are with him.

By Katherine M. Hafner in New York

Media interest in *Mathematica*

A Top Scientist's Latest: Math Software

By ANDREW POLLACK

Special to The New York Times

SANTA CLARA, Calif., June 23 — A man widely regarded as one of the world's most brilliant scientists formally entered the computer business today with a program intended to do for mathematics what the calculator did for arithmetic.

Stephen Wolfram, who earned a Ph.D. in physics when he was 20, is the force behind the new program, Mathematica, which seems to be a dream come true for math students who have trouble factoring complex polynomials, graphing elliptical functions or calculating pi to 2,000 decimal places.

Mathematica, which is also intended for use by scientists and engineers, can solve equations in algebra and calculus and draw two- and three-dimensional graphs instantly.

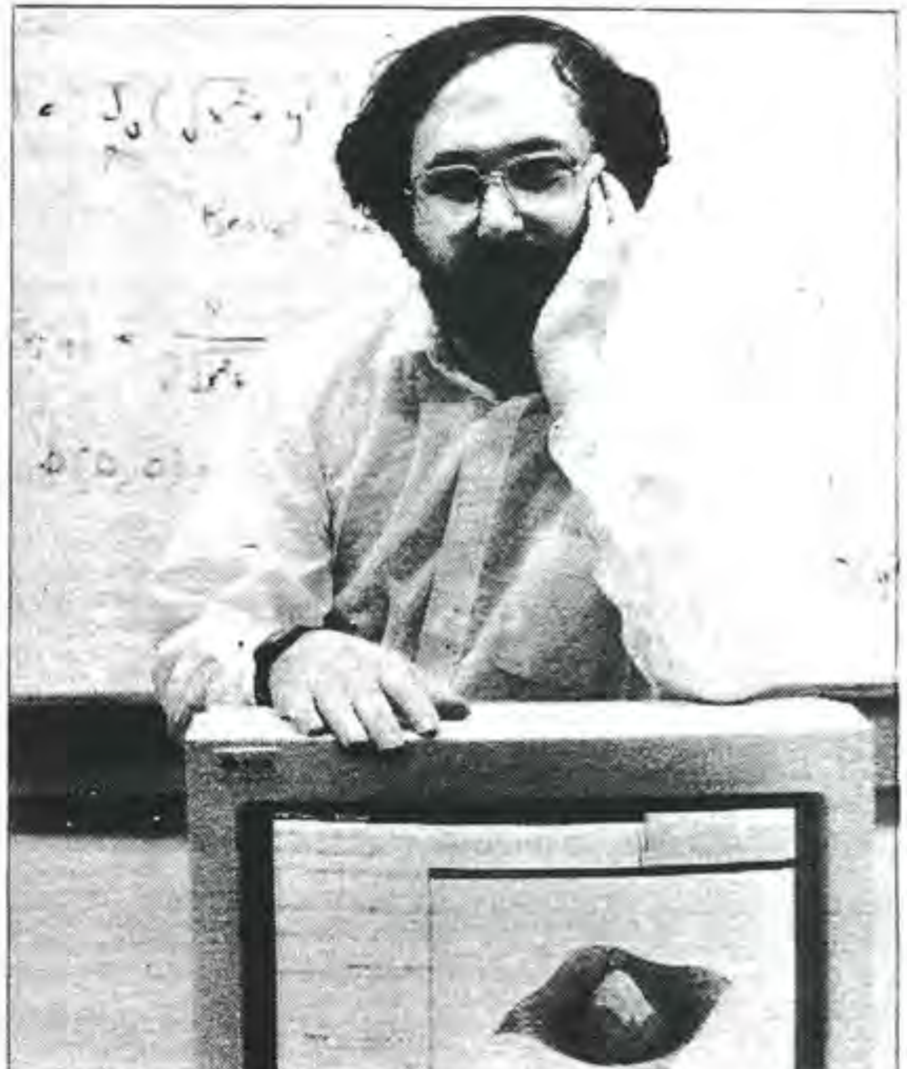
Math Done the Old Way

Dr. Wolfram, who is 28 years old, said that, surprisingly, mathematics is still done largely with pencil, paper and calculator.

Whether Dr. Wolfram, a professor at the University of Illinois, proves to be as good an entrepreneur as he is a scientist remains to be seen. His program is not the first directed at mathematics, and many previous ones have not been great commercial successes.

Dr. Wolfram's program has attracted unusual attention, partly because of who he is and partly because of the companies that are backing

Continued on Page 30



The New York Times/Terrence McCarthy

Stephen Wolfram, with a frame from his software, Mathematica, that shows a three-dimensional plot of the wave pattern on a drum head.

Twilight of the pencil?

William H. Press

Mathematica. *Wolfram Research Inc., PO Box 6059, Champaign, Illinois 61821. $495 (Macintosh Plus/SE version); $795 (Macintosh II version). Stop Press: on 15 December a version becomes available for 386-based IBM-PCs; prices from $695 to $1,295.*
Mathematica: A System of Doing Mathematics by Computer. By Stephen Wolfram. *Addison-Wesley: 1988. Pp. 749. Hbk $40.95; £36.85; pbk $27.75, £19.95.*

Mathematica is a remarkable, perhaps even revolutionary, computer program written by a small group of young programmers at Wolfram Research Inc. Inside it are of the order of 200,000 lines of source code. More interesting is that *behind* the program there are a number of powerful, perhaps radical, ideas on what amounts to 'machine-assisted human thought' as it applies to mathematics. The manual, available in book form, provides an intellectual entrée to these ideas; the software is separately available from Wolfram Research (for Macintosh machines) and from Sun Microsystems Inc. (for Sun workstations). *Mathematica* is included free with the new Next computers.

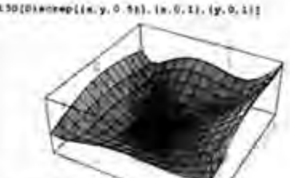
The program is sufficiently different from existing computer tools, and so powerful in its way, that it is hard to describe exactly what it is. The easiest description, though it misses the mark, is to say that it is a symbolic manipulation package comparable, say, to *Macsyma* (Symbolics Inc.). If you type 'Integrate [1/1+x 6]', *Mathematica* echoes back the indefinite integral, a complicated expression involving arctangents and logarithms. As compared with *Macsyma* (available for VAX, Sun and Apollo workstations, but not for the Macintosh), *Mathematica* is less good at doing integrals, but seems better at solving systems of (even nonlinear) equations, number theoretical calculations and special functions. *Mathematica* has the vastly superior user interface, but its output format (like *Macsyma*'s) leaves something to be desired.

The reason that this comparison misses the mark is that *Mathematica* is deeper in its concept than previous symbolic mathematics programs. It is a kind of 'general algorithmic engine', for which symbolic mathematics is only one possible application. The engine can operate with equal facility on algebraic objects, numerical objects (arbitrary precision in integer or floating), graphical objects (represented ultimately in *Postscript* graphics language), or any other object that the user simply defines 'on the fly'. The interface among these modes is practically seamless.

If one thinks of *Mathematica* as a computer language — another approximate but incomplete description — then it is something like LISP, but with a friendly user interface, with several dozen pre-defined data structures, and with several hundred pre-defined operations on these structures. Also pre-defined are anyone's favourite control structures, an inclusive set taken from C, FORTRAN and Pascal. *Mathematica* emerges as a higher-level language than any of its antecedents. It is a huge language, in terms of its number of pre-defined operations.

All of this structure means that *Mathematica* can be used by the novice as a kind of super-BASIC. Follow the examples in the book, and you have: (i) a numerical calculator that knows lots of special functions, handles real or complex numbers in arbitrary precision; or (ii) a graphics package that plots functions, contour plots, shaded three-dimensional surfaces (in colour if you like); or (iii) a symbolic mathematics package that can do algebra, matrix algebra, derivatives, integrals and so forth.

```
Do[ p[4i+2j+k+1] = {i,j,k}, {i,0,1},{j,0,1},{k,0,1}]
Norm[p_] := Sqrt[p[[1]]^2+p[[2]]^2+p[[3]]^2]
w[i_][q_] := Product[1-Abs[q[[j]]-p[i][[j]]], {j,3}]
Force[p1_,p2_] := (p1-p2)/Norm[p1-p2]^3
GridForce[q_] := Sum[ w[i][q] Force[p[i],ReceivePt], {i,8}]
ReceivePt = {2.6,1.9,1.0};
Discrep[q_] := Block[ {F1,F2},
                      F1 = Force[q,ReceivePt];
                      F2 = GridForce[q];
                      Norm[F1-F2]/Norm[F1]
                    ]
Plot3D[Discrep[{x,y,0.5}],{x,0,1},{y,0,1}]
```



```
Discrep[{1/2,1/2,1/2}]
ReceivePt = {0,1/2,1/2};
Discrep[{1/2,1/2,1/2}]//InputForm
```

Using Mathematica *for algorithmic experimentation.*

A nice feature is the way that your session is recorded in an on-screen 'Notebook', in which the program statements can easily be cleaned up, explanatory text inserted and the result printed out in laser-printer fonts as an elegant-looking document. Wolfram's informal statistics suggest that a surprisingly large number of users are financial analysts, who are probably using *Mathematica* in this mode.

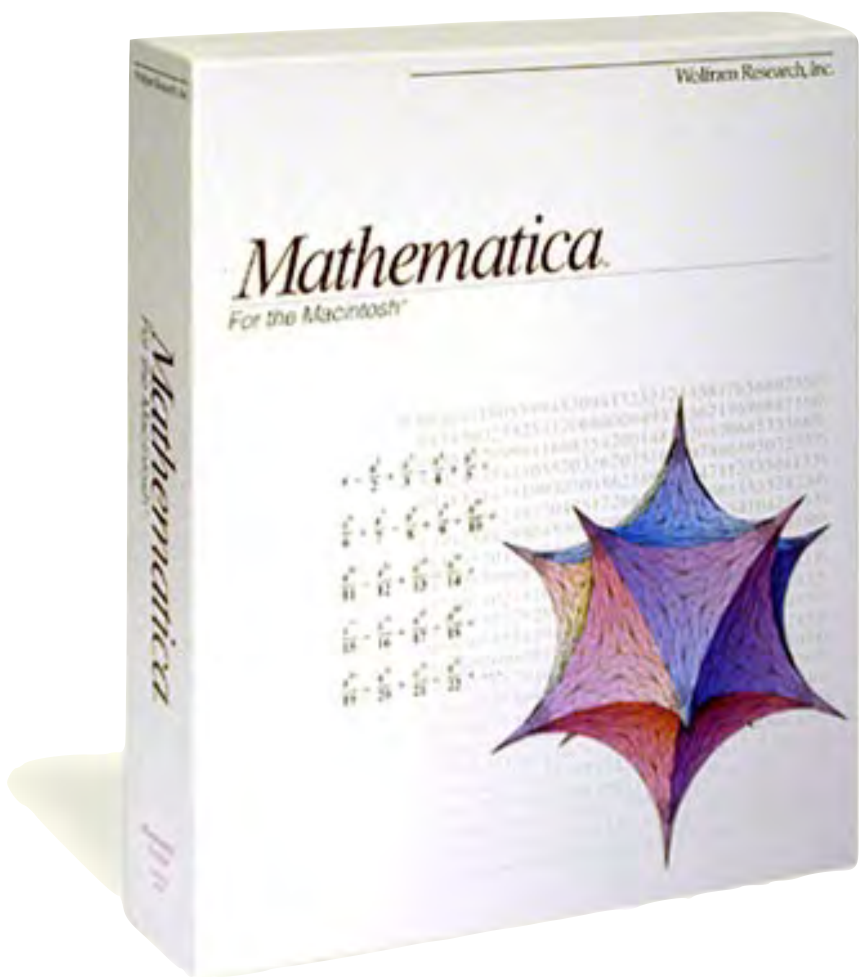
This is flashy stuff, but by itself it is unlikely to revolutionize the way that working scientists do mathematics. Two deeper aspects of the program — which require a correspondingly greater investment of effort to master — might, however, bring about just such a change.

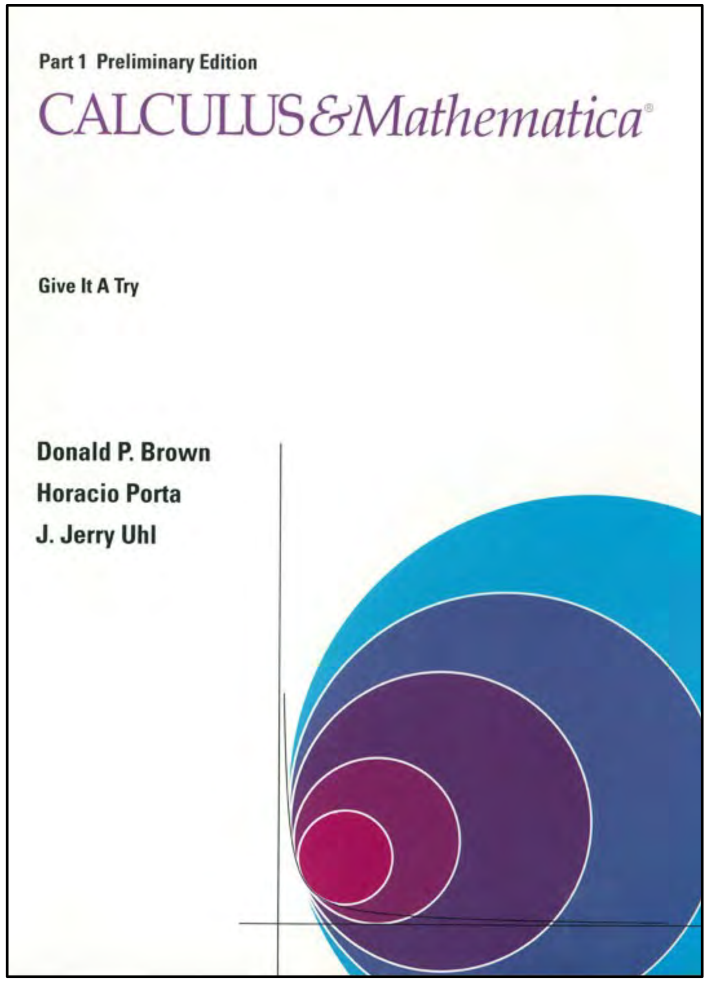
The first of these is *Mathematica*'s usefulness as a concise test-bed for algorithmic experimentation. Some sense of this may be conveyed in the accompanying figure, even without detailed explanation. A sequence of definitions (lines with the ':=' construction) creates eight 3-vectors on the corner of a unit cube, defines 'triangular' linear interpolation functions, and teaches *Mathematica* to compare the inverse square force resulting from a particle at an arbitrary location with that from the particle's interpolation onto the corners of the cube. The definitions are then used in three ways: first, to make plot of the force comparison on a two-dimensional slice through the unit cube; second, to compute a particular value; and third, to give an algebraic formula. It would be difficult, if not impossible, to get the same intuitive grasp of a problem like this by hand calculation.

Notice that nothing in the definitions specifies whether they are numerical or symbolic. That is determined, interpretatively, by the actual calling arguments in each invocation. This, in fact, relates to the second deep aspect of *Mathematica*. It has a sophisticated hierarchical mechanism for 'overloading' procedural definitions, even its most fundamental pre-defined ones. The same operation (addition, for example, or 'do loop') can be defined to work on numbers, vectors or any other object, in any desired way. Functions automatically parse their arguments and search for a previously stored definition that matches syntactically. If nothing matches, then the function becomes a new symbolic 'atom'. Once one masters the knack, one can 'teach' *Mathematica* to be anything one wants it to be, or to deal with virtually any kind of mathematical object.

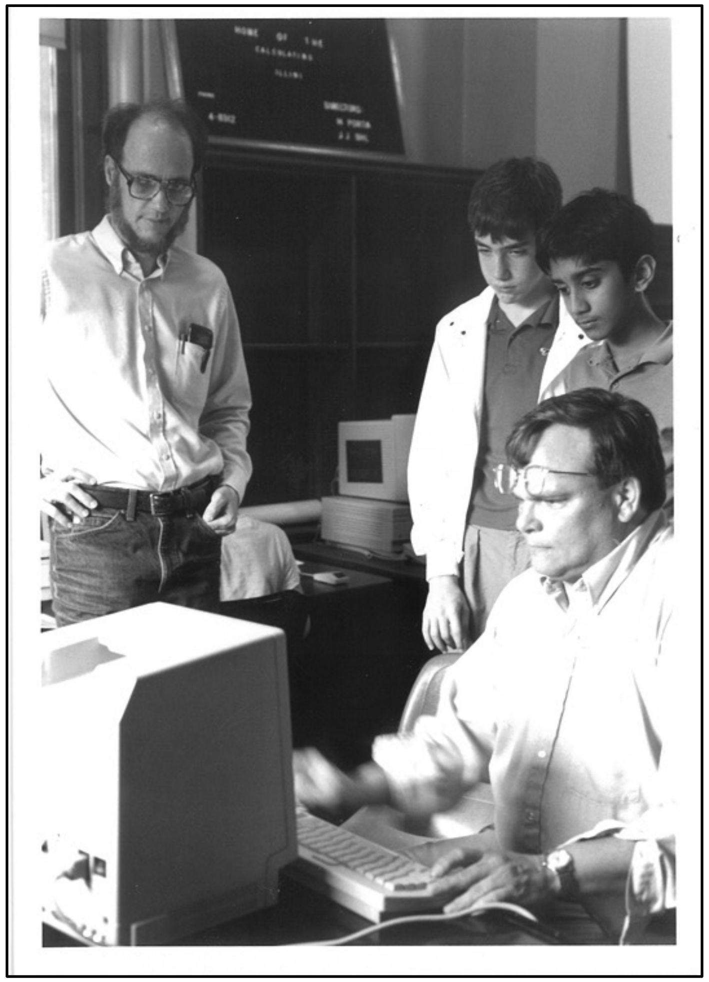
The potential user should be aware of some practicalities. *Mathematica* requires at least four megabytes of memory to run effectively on a Macintosh — and Mac

Teaching calculus with *Mathematica*



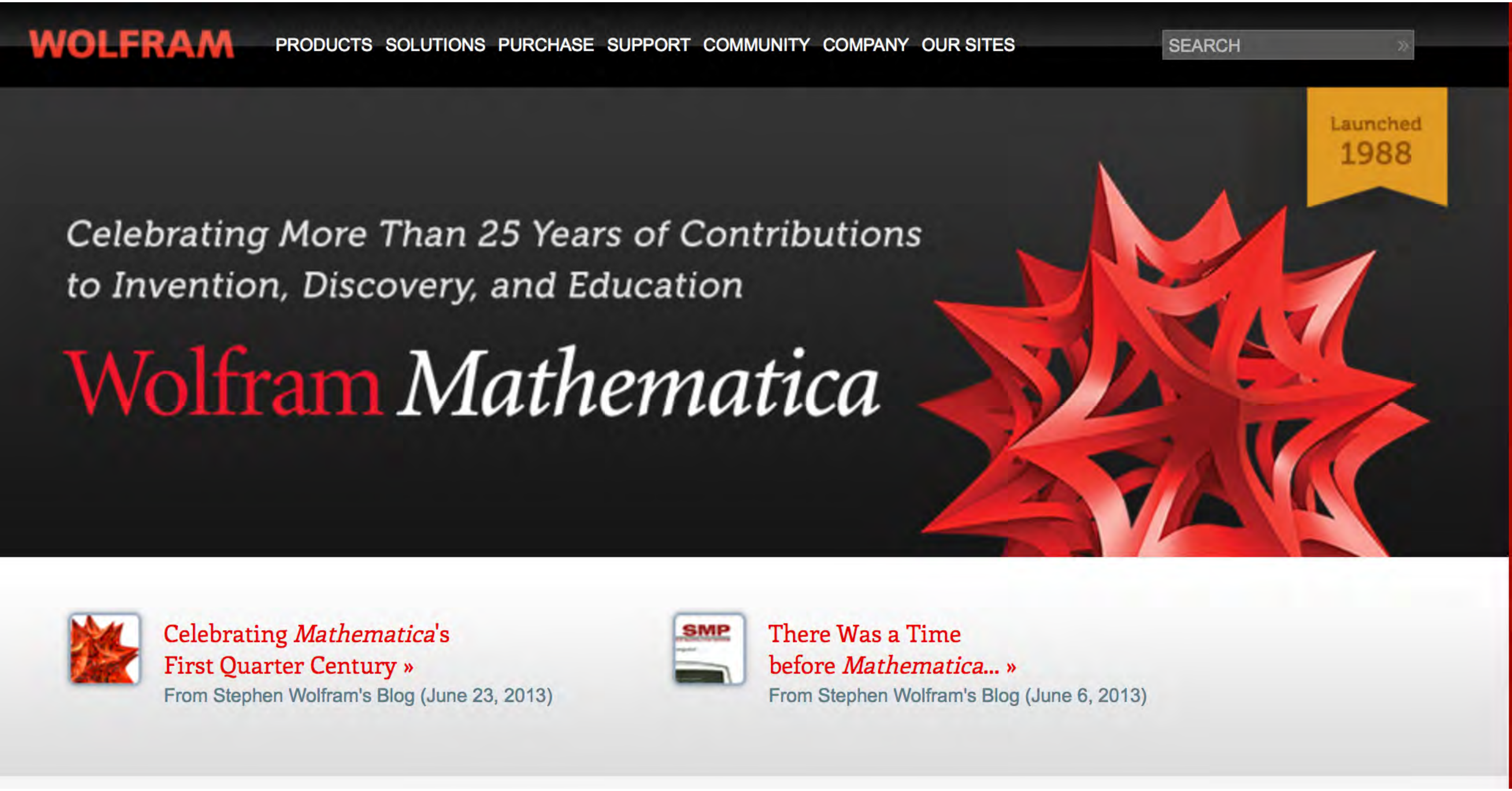


First *Mathematica* based calculus course

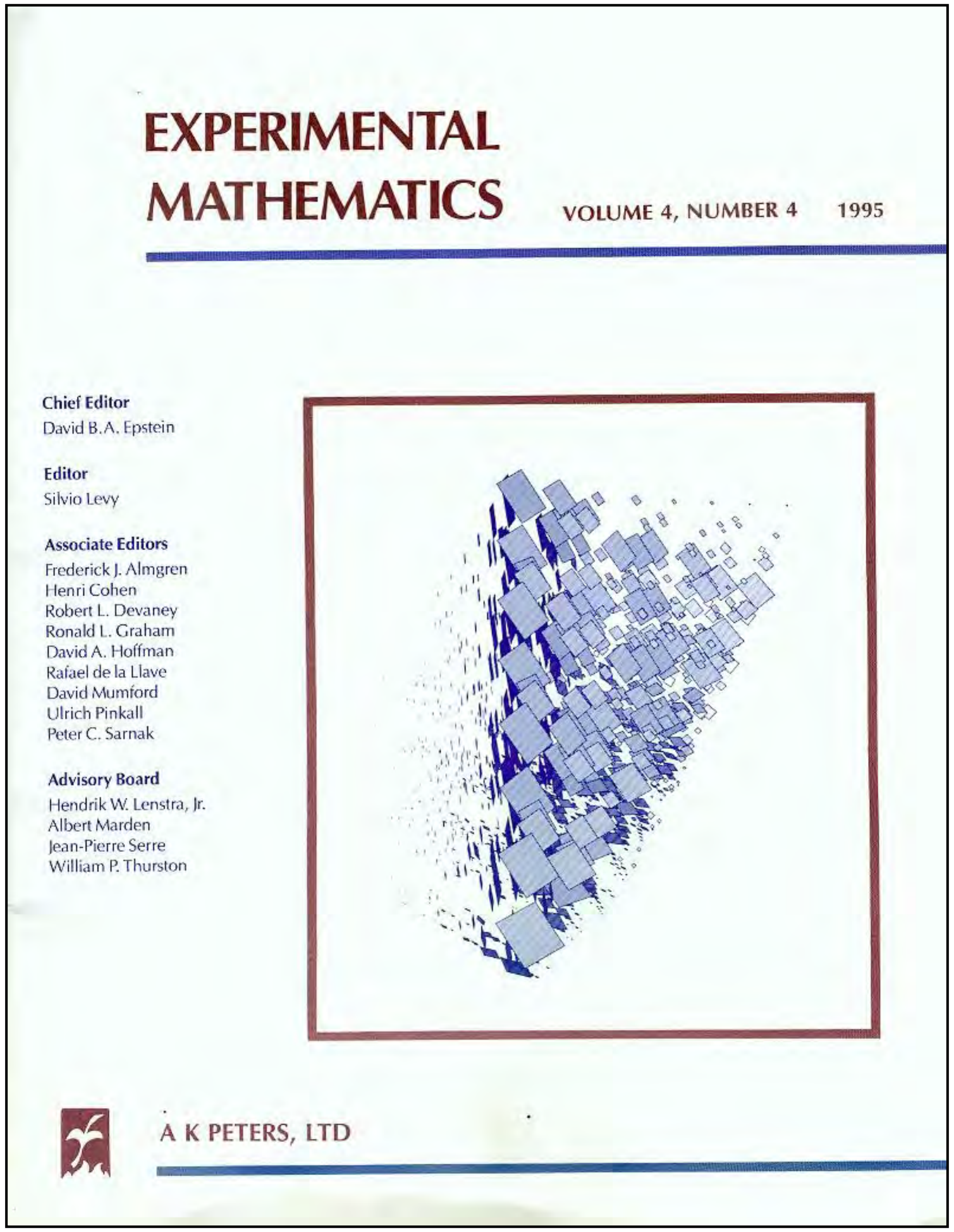


Jerry Uhl and calculus students

http://www.mathematica25.com



Embracing the computer: Experimental Mathematics

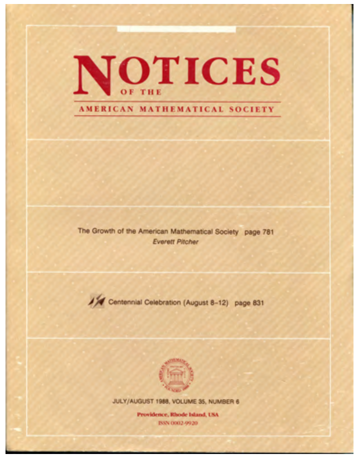


Experimental Mathematics journal established by David Epstein, Silvio Levy, and Klaus Peters.

"Experimental mathematics" is an approach to mathematics in which computation is used to investigate mathematical objects and identify properties and patterns.

Founders stated goal: The objective is to play a role in the **discovery** of formal proofs, **not to displace** them.

First issue published in 1992.



Oct 1992: The Borweins in "Computers and Mathematics"

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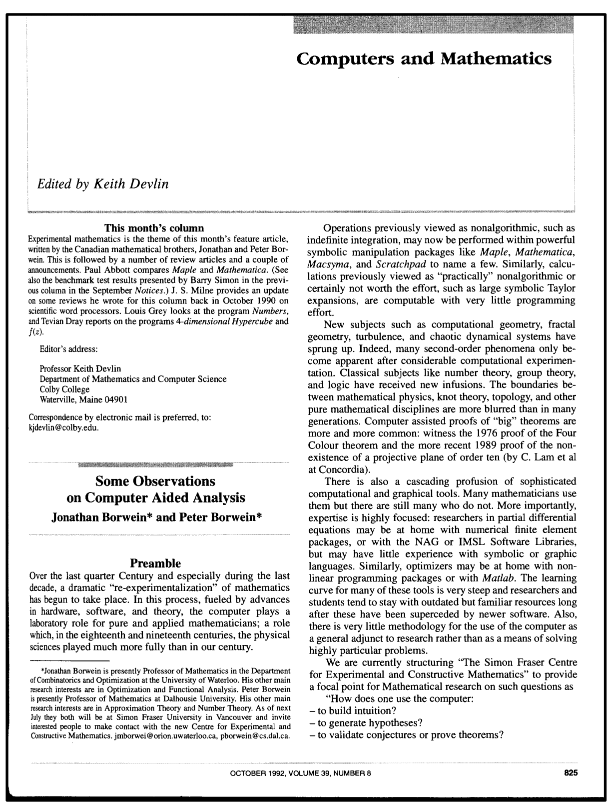
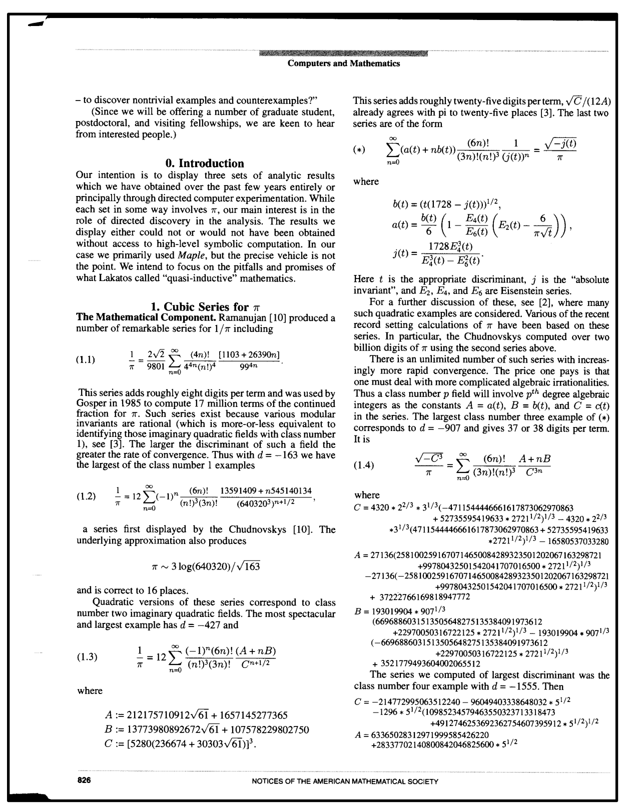
Experimental mathematics is the theme of this month's feature article, written by the Canadian mathematical brothers, Jonathan and Peter Borwein. This is followed by a number of review articles and a couple of announcements. Paul Abbott compares *Maple* and *Mathematica*. (See also the benchmark test results presented by Barry Simon in the previous column in the September *Notices*.) J. S. Milne provides an update on some reviews he wrote for this column back in October 1990 on scientific word processors. Louis Grey looks at the program *Numbers*, and Tevian Dray reports on the programs 4-*dimensional Hypercube* and





Some Observations on Computer Aided Analysis

Jonathan Borwein* and Peter Borwein*

Preamble

Over the last quarter Century and especially during the last decade, a dramatic "re-experimentalization" of mathematics has begun to take place. In this process, fueled by advances in hardware, software, and theory, the computer plays a laboratory role for pure and applied mathematicians; a role which, in the eighteenth and nineteenth centuries, the physical sciences played much more fully than in our century.

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*Jonathan Borwein is presently Professor of Mathematics in the Department of Combinatorics and Optimization at the University of Waterloo. His other main research interests are in Optimization and Functional Analysis. Peter Borwein is presently Professor of Mathematics at Dalhousie University. His other main research interests are in Approximation Theory and Number Theory. As of next July they both will be at Simon Fraser University in Vancouver and invite interested people to make contact with the new Centre for Experimental and Constructive Mathematics. jmborwei@orion.uwaterloo.ca, pborwein@cs.dal.ca.

Oct 1992: The Borweins in "Computers and Mathematics"

Computers and Mathematics

Edited by Keith Devlin

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Some Observations on Computer Aided Analysis

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Operations previously viewed as nonalgorithmic, such as indefinite integration, may now be performed within powerful symbolic manipulation packages like *Maple*, *Mathematica*, *Macsyma*, and *Scratchpad* to name a few. Similarly, calculations previously viewed as "practically" nonalgorithmic or certainly not worth the effort, such as large symbolic Taylor expansions, are computable with very little programming effort.

New subjects such as computational geometry, fractal geometry, turbulence, and chaotic dynamical systems have sprung up. Indeed, many second-order phenomena only become apparent after considerable computational experimentation. Classical subjects like number theory, group theory, and logic have received new infusions. The boundaries between mathematical physics, knot theory, topology, and other pure mathematical disciplines are more blurred than in many generations. Computer assisted proofs of "big" theorems are more and more common: witness the 1976 proof of the Four Colour theorem and the more recent 1989 proof of the non-existence of a projective plane of order ten (by C. Lam et al at Concordia).

There is also a cascading profusion of sophisticated computational and graphical tools. Many mathematicians use them but there are still many who do not. More importantly, expertise is highly focused: researchers in partial differential equations may be at home with numerical finite element packages, or with the NAG or IMSL Software Libraries, but may have little experience with symbolic or graphic languages. Similarly, optimizers may be at home with non-linear programming packages or with *Matlab*. The learning curve for many of these tools is very steep and researchers and students tend to stay with outdated but familiar resources long after these have been superceded by newer software. Also, there is very little methodology for the use of the computer as a general adjunct to research rather than as a means of solving highly particular problems.

We are currently structuring "The Simon Fraser Centre for Experimental and Constructive Mathematics" to provide a focal point for Mathematical research on such questions as

"How does one use the computer:
– to build intuition?
– to generate hypotheses?
– to validate conjectures or prove theorems?

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– to discover nontrivial examples and counterexamples?"

(Since we will be offering a number of graduate student, postdoctoral, and visiting fellowships, we are keen to hear from interested people.)

0. Introduction

Our intention is to display three sets of analytic results which we have obtained over the past few years entirely or principally through directed computer experimentation. While each set in some way involves π, our main interest is in the role of directed discovery in the analysis. The results we display either could not or would not have been obtained without access to high-level symbolic computation. In our case we primarily used *Maple*, but the precise vehicle is not the point. We intend to focus on the pitfalls and promises of what Lakatos called "quasi-inductive" mathematics.

1. Cubic Series for π

The Mathematical Component. Ramanujan [10] produced a number of remarkable series for $1/\pi$ including

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}\frac{[1103+26390n]}{99^{4n}}. \tag{1.1}$$

This series adds roughly eight digits per term and was used by Gosper in 1985 to compute 17 million terms of the continued fraction for π. Such series exist because various modular invariants are rational (which is more-or-less equivalent to identifying those imaginary quadratic fields with class number 1), see [3]. The larger the discriminant of such a field the greater the rate of convergence. Thus with $d=-163$ we have the largest of the class number 1 examples

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}(-1)^n\frac{(6n)!}{(n!)^3(3n)!}\frac{13591409+n545140134}{(640320^3)^{n+1/2}}, \tag{1.2}$$

a series first displayed by the Chudnovskys [10]. The underlying approximation also produces

$$\pi \sim 3\log(640320)/\sqrt{163}$$

and is correct to 16 places.

Quadratic versions of these series correspond to class number two imaginary quadratic fields. The most spectacular and largest example has $d=-427$ and

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!}{(n!)^3(3n)!}\frac{(A+nB)}{C^{n+1/2}} \tag{1.3}$$

where

$$A := 212175710912\sqrt{61} + 1657145277365$$
$$B := 13773980892672\sqrt{61} + 107578229802750$$
$$C := [5280(236674 + 30303\sqrt{61})]^3.$$

This series adds roughly twenty-five digits per term, $\sqrt{C}/(12A)$ already agrees with pi to twenty-five places [3]. The last two series are of the form

$$\sum_{n=0}^{\infty}(a(t)+nb(t))\frac{(6n)!}{(3n)!(n!)^3}\frac{1}{(j(t))^n} = \frac{\sqrt{-j(t)}}{\pi} \tag{$*$}$$

where

$$b(t) = (t(1728 - j(t)))^{1/2},$$
$$a(t) = \frac{b(t)}{6}\left(1 - \frac{E_4(t)}{E_6(t)}\left(E_2(t) - \frac{6}{\pi\sqrt{t}}\right)\right),$$
$$j(t) = \frac{1728E_4^3(t)}{E_4^3(t) - E_6^2(t)}.$$

Here t is the appropriate discriminant, j is the "absolute invariant", and E_2, E_4, and E_6 are Eisenstein series.

For a further discussion of these, see [2], where many such quadratic examples are considered. Various of the recent record setting calculations of π have been based on these series. In particular, the Chudnovskys computed over two billion digits of π using the second series above.

There is an unlimited number of such series with increasingly more rapid convergence. The price one pays is that one must deal with more complicated algebraic irrationalities. Thus a class number p field will involve p^{th} degree algebraic integers as the constants $A=a(t)$, $B=b(t)$, and $C=c(t)$ in the series. The largest class number three example of ($*$) corresponds to $d=-907$ and gives 37 or 38 digits per term. It is

$$\frac{\sqrt{-C^3}}{\pi} = \sum_{n=0}^{\infty}\frac{(6n)!}{(3n)!(n!)^3}\frac{A+nB}{C^{3n}} \tag{1.4}$$

where

$$C = 4320 * 2^{2/3} * 3^{1/3}(-4711544446661617873062970863 + 52735595419633 * 2721^{1/2})^{1/3} - 4320 * 2^{2/3} * 3^{1/3}(4711544446661617873062970863 + 52735595419633 * 2721^{1/2})^{1/3} - 16580537033280$$

$$A = 27136(258100259167071465008428932350120206716329872 1 + 9978043250154204170701 6500 * 2721^{1/2})^{1/3} - 27136(-2581002591670714650084289323501202067163298721 + 99780432501542041707016500 * 2721^{1/2})^{1/3} + 37222766169818947772$$

$$B = 193019904 * 907^{1/3}(669688603151350564827513538409197361 2 + 22970050316722125 * 2721^{1/2})^{1/3} - 193019904 * 907^{1/3}(-6696886031513505648275135384091973612 + 22970050316722125 * 2721^{1/2})^{1/3} + 352177949360400206551 2$$

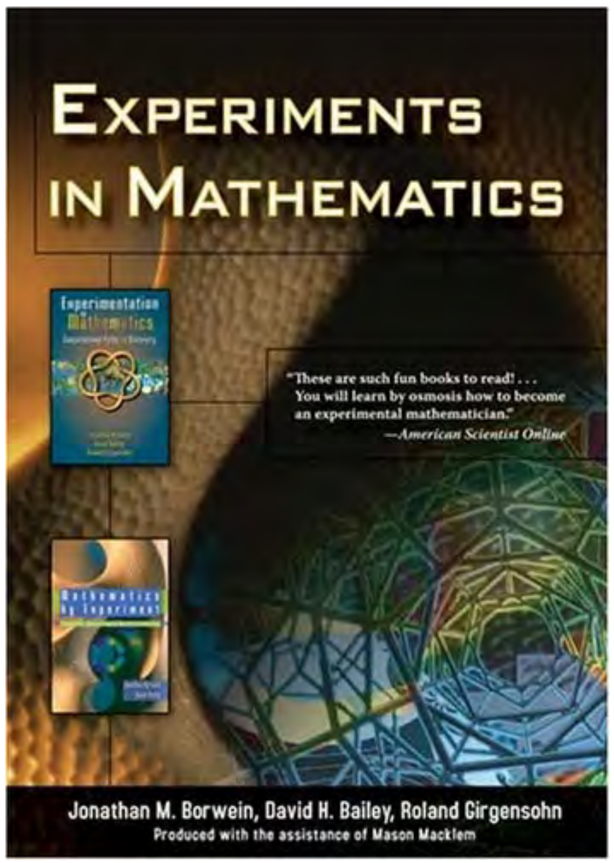
The series we computed of largest discriminant was the class number four example with $d=-1555$. Then

$$C = -214772995063512240 - 96049403338648032 * 5^{1/2} - 1296 * 5^{1/2}(10985234579463550323713318473 + 4912746253692362754607395912 * 5^{1/2})^{1/2}$$

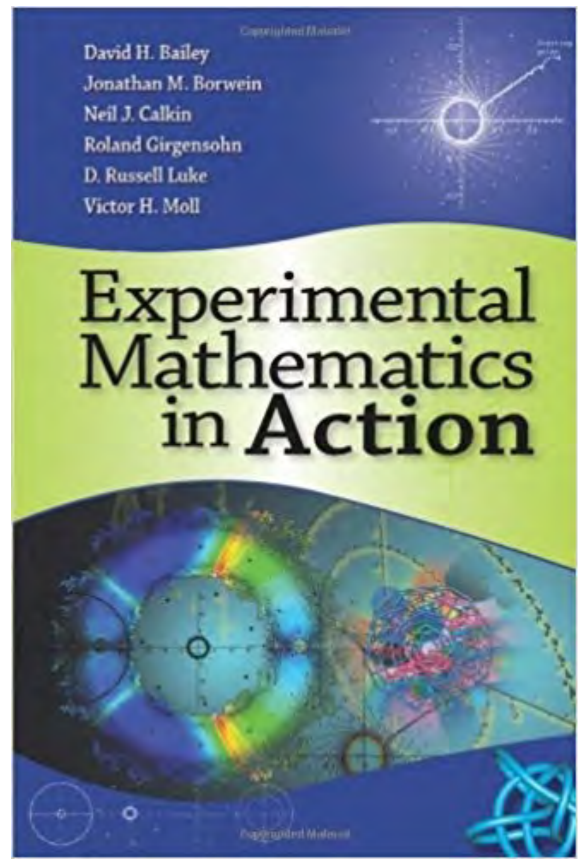
$$A = 63365028312971999585426220 + 28337702140800842046825600 * 5^{1/2}$$

Embracing the computer: Experimental Mathematics

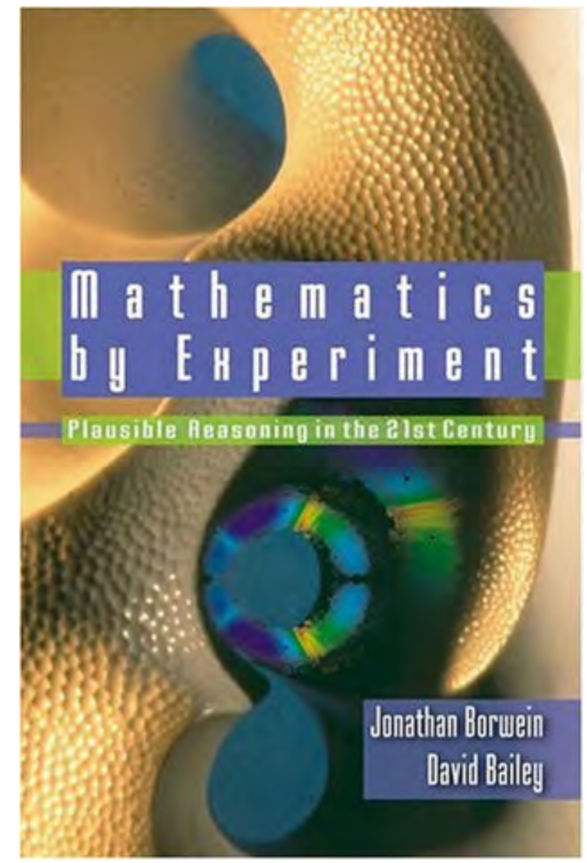
Borwein, Bailey, *et al*



2004

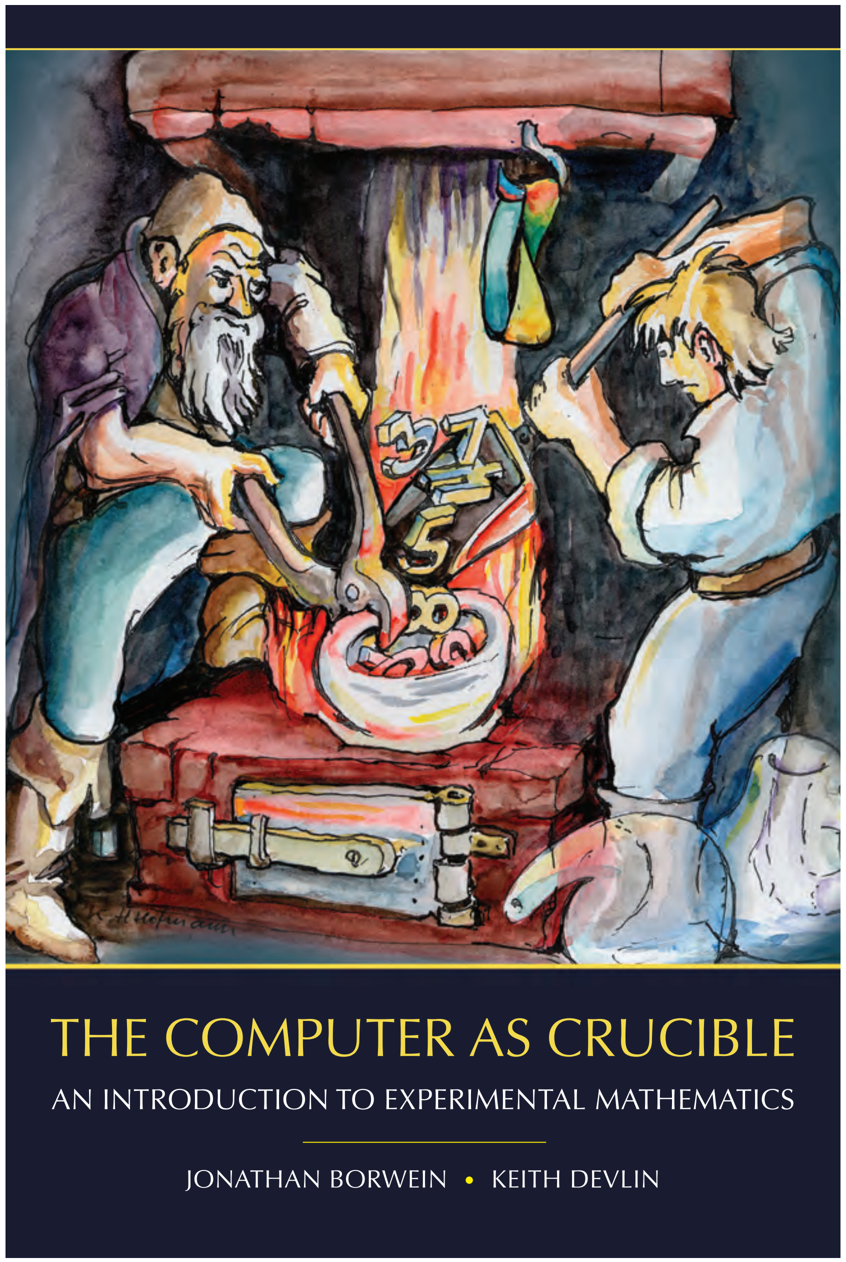


2007



2008

Embracing the computer: Experimental Mathematics



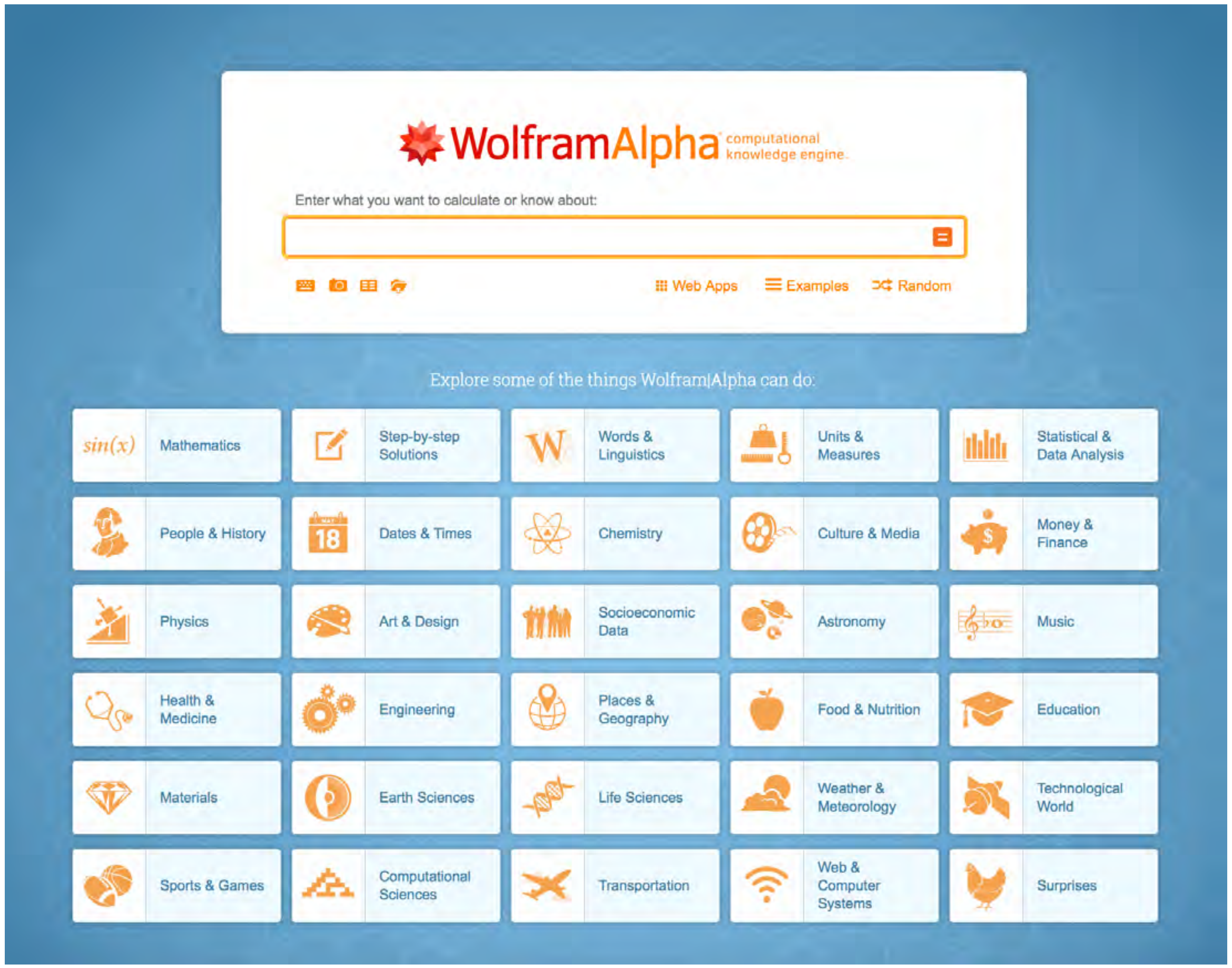
AK Peters/CRC Press, 2008

THINK – COMPUTE – REPEAT

Some techniques of experimental mathematics

1. Symbolic computation using a computer algebra system such as *Mathematica* or *Maple*
2. Data visualization methods
3. Search Web resources, eg. Sloane's *Online Encyclopedia of Integer Sequences*
4. Integer-relation methods, such as the PSLQ algorithm
5. High-precision integer and floating-point arithmetic
6. High-precision numerical evaluation of integrals and summation of infinite series,
7. Use of the Wilf-Zeilberger algorithm for proving summation identities
8. Iterative approximations to continuous functions
9. Identification of functions based on graph characteristics

2009: Wolfram Alpha



Keith Devlin, Contributor
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All The Mathematical Methods I Learned In My University Math Degree Became Obsolete In My Lifetime

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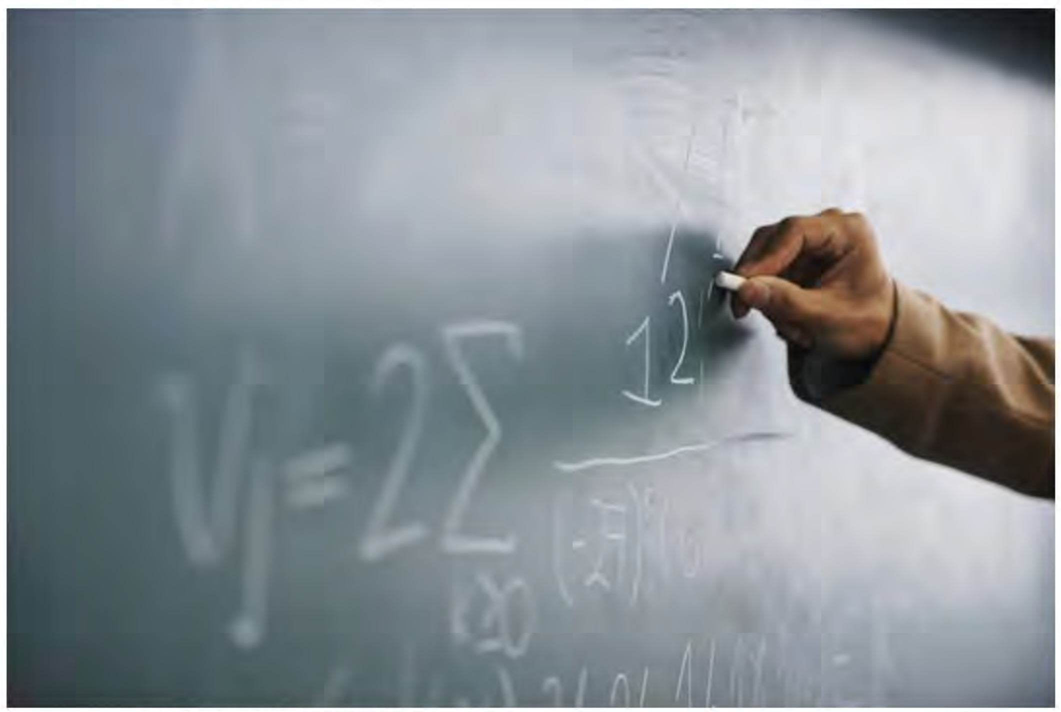

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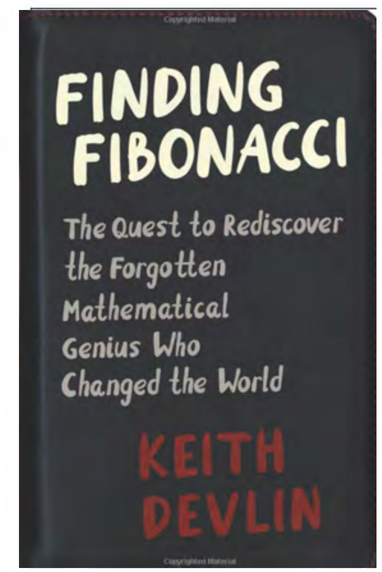

Buy it now

If you are connected with the world of K-12 mathematics education, it's highly unlikely that a day will go by without you uttering, writing, hearing, or reading the term "number sense". In contrast everyone else on the planet would be hard pressed to describe what it is. Though entering the term into Google will return close to 38 million hits, it has yet to enter the world's collective consciousness. Stanford mathematician Keith Devlin explains what it is.